- $f : \mathbb{R}^n \to \mathbb{R}$ differentiable in x_0

  $\nabla f(x_0) = df(x_0) = \left(\frac{\partial f}{\partial x^1}(x_0), \ldots \frac{\partial f}{\partial x^n}(x_0) \right)$ is the Matrix

  repn. of the linear map $df : \mathbb{R}^n \to \mathbb{R}$

- If for given $f : \mathbb{R}^n \to \mathbb{R}$, $\frac{\partial f}{\partial x^i}$ exist and continous

  for all $i = 1, \ldots, n$ then f is differentiable

Differentiation rules

$f, g : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad f, g$ diff at $x_0 \in \Omega$. Then $f+g$, $f \cdot g$ are differentiable.

1) $d(f+g)(x_0) = df(x_0) + dg(x_0)$

2) $d(f \cdot g)(x_0) = (df(x_0)) \cdot g(x_0) + f(x_0) \cdot dg(x_0)$

3) if $g(x_0) \neq 0$ then f/g is diff and

$$d\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0) \cdot df(x_0) - f(x_0)\, dg(x_0)}{g(x_0)^2}$$

Chain rule

I $f : \mathbb{R}^n \to \mathbb{R}$ diff in x_0, $g : \mathbb{R} \to \mathbb{R}$ diff in $f(x_0)$

Then $g \circ f : \mathbb{R}^n \to \mathbb{R}$ is diff in x_0 and

$$d(g \circ f)(x_0) = dg(f(x_0)) . df(x_0) = g'(f(x_0)) . df(x_0)$$

II) $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}^n$, $t \to (g_1(t), \dots g_n(t))$, g is diff in t_0 and f is diff in $g(t_0)$. Then $f \circ g : \mathbb{R} \to \mathbb{R}$ is diff in t_0

and

$$\frac{d}{dt}(f \circ g)(t_0) = df(g(t_0)) . g'(t_0)$$

$$= \nabla f(g(t_0)) \cdot (g_1'(t_0), \dots, g_n'(t_0))$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(g(t_0)) \frac{dg_i}{dt}(t_0)$$

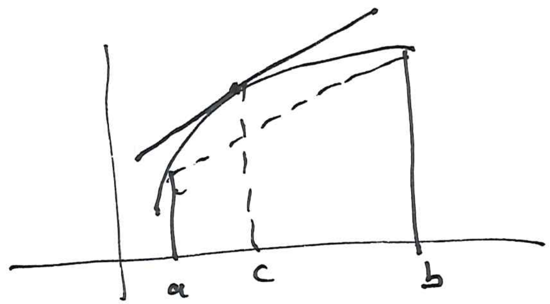
Applications of the chain rule.

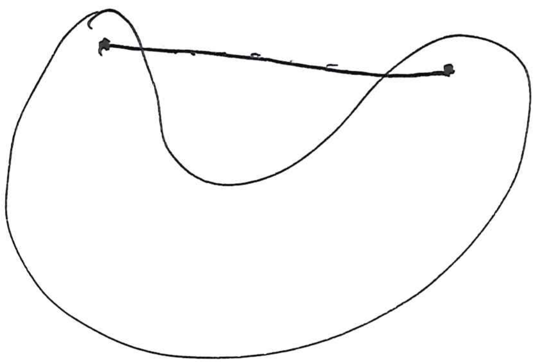
① Mean value theorem in several variables.

Recall: For $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ differentiable,

$\exists\ c \in [a,b] \subset \Omega$ such that

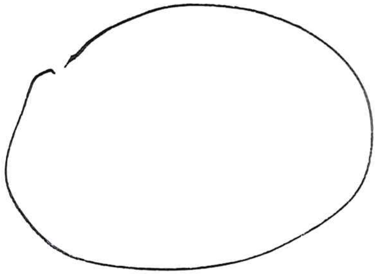
$$f'(c)(b-a) = f(b) - f(a).$$





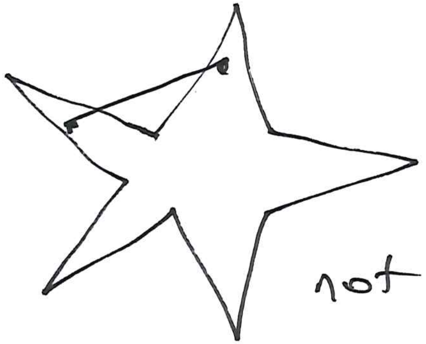
$\Omega \subset \mathbb{R}^2$

Defn. A set $K \subseteq \mathbb{R}^n$ is called convex if for every pair of points $x, y \in K$, the line segment between x, y; $(1-t)x + ty$ $t \in [0,1]$, is also in K.



convex.

convex



not convex

<u>lemma</u> (Bsp 7.2.2(ii)). let $\Omega \subset \mathbb{R}^n$ convex $f : \Omega \to \mathbb{R}$ differentiable function. let $x_0, x_1 \in \Omega$ and $x_t := (1-t)\, x_0 + t\, x_1$, $t \in [0,1]$, $x_t \in \Omega$.

Then $\exists$ $\theta \in [0,1]$ such that

$$f(x_1) - f(x_0) = df(x_\theta) \cdot (x_1 - x_0). \quad (*)$$

ie $\exists$ a pt x_θ on the line segment between x_0 and x_1 such that $(*)$ holds.

Pf. Let $g(t) := (1-t)x_0 + tx_1 = x_t$ $\qquad g'(t) = -x_0 + x_1 = x_1 - x_0$

Then $(f \circ g)(t) : \mathbb{R} \to \mathbb{R}$, $t \mapsto f(g(t))$ is differentiable

for $t \in [0,1]$. Hence by 1 dim'l mean value theorem

$\exists\ \theta \in [0,1]$ such that

$$(f \circ g)(1) - (f \circ g)(0) = (f \circ g)'(\theta)(1-0).$$

$$\underbrace{f(g(1))}_{f(x_1)} - \underbrace{f(g(0))}_{f(x_0)} = (df)(g(\theta)) \cdot \underbrace{g'(\theta)}_{(x_1 - x_0)}.$$

$$\boxed{f(x_1) - f(x_0) = (df)(x_\theta) \cdot (x_1 - x_0)}$$

$\boxtimes$

Cor. Let Ω be open convex set. f differentiable.

If $\nabla f(x) = 0 \quad \forall x \in \Omega$, then f is constant.

Pf. Let $a, b \in \Omega$. Then by MVT $\exists\, c$ on the line segment between a, b such that

$$f(b) - f(a) = \nabla f(c) \cdot (b-a).$$

Since $\nabla f(x) = 0 \quad \forall x \in \Omega$

$$f(b) - f(a) = 0 \cdot (b-a) = 0 \Rightarrow f(b) = f(a)$$

a, b was arbitrary so f is constant.

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^s \, \frac{dt}{t}.$$

Thm. Let $h(s,t)$ be a continuously differentiable function of 2 variables and $b(t)$ differentiable function of one variable. Then the function

$$u(t) := \int_a^{b(t)} h(s,t)\, ds \quad , \text{ where it is defined,}$$

is differentiable with derivative

$$u'(t) = h(b(t), t) \cdot b'(t) + \int_a^{b(t)} \frac{\partial h}{\partial t}(s,t)\, ds.$$

Cor 1 $\quad u(t) := \int_0^t h(s,t)\,ds$

then $\quad u'(t) = h(t,t) + \int_0^t \frac{\partial h}{\partial t}(s,t)\,ds$

2 $\quad u(t) := \int_a^b h(s,t)\,ds$ then

$$u'(t) = \int_a^b \frac{\partial h}{\partial t}(s,t)\,ds.$$

Idea $\quad f(x,y) = \int_a^x h(s,y)\,ds$

$$g(t) = \begin{pmatrix} b(t) \\ t \end{pmatrix}$$

$u = f \circ g$.

Ex. $\displaystyle\int_0^1 \frac{x^5-1}{\log x}\,dx$

The indefinite integral $\displaystyle\int \frac{x^5-1}{\log x}\,dx$ is not computable in terms of elementary functions.

let $\displaystyle u(\alpha) := \int_0^1 \frac{x^\alpha-1}{\log x}\,dx.$ $\qquad u(5) = ?$

For $\alpha \geqq 0$ $u(\alpha)$ satisfies the assumptions in the previous theorem

$$h(\alpha, x) := \frac{x^\alpha - 1}{\log x} \qquad \frac{\partial h}{\partial \alpha} = \frac{1}{\log x}\, x^\alpha \log x = x^\alpha.$$

Using the theorem $u'(\alpha) = \int_0^1 \underbrace{\frac{\partial}{\partial\alpha}\left(\frac{x^\alpha - 1}{\log x}\right)}_{x^\alpha} dx$

$$= \int_0^1 x^\alpha \, dx = \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_0^1 = \frac{1}{\alpha+1}$$

$$u'(\alpha) = \frac{1}{\alpha+1}$$

$$\Longrightarrow \quad u(\alpha) = \int \frac{1}{\alpha+1} \, d\alpha \Rightarrow \log(\alpha+1) + c$$

for a constant c that we still have to determine.

But $u(0) = \int_0^1 \frac{x^0 - 1}{\log x} dx = 0$. In

$u(\alpha) = \log(\alpha+1) + c$ set $\alpha = 0 \Rightarrow u(0) = 0 = \log 1 + c$
$\Rightarrow c = 0$.

$u(\alpha) := \int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(\alpha+1)$. In partich $\boxed{u(5) = \log 6}$.

Recall: If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

diff in $x_0 \in \Omega$

Then $\nabla f(x_0)$ gives the direction

and $\|\nabla f(x_0)\|$ gives the magnitude of

steepest ascent of f at the point x_0.

let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^1-function
ie diff. w/ continous partial derivatives.

let $c \in \mathbb{R}$. let

$$L(c) := f^{-1}(\{c\}) = \{x \in \mathbb{R}^n \mid f(x) = c, x \in \Omega\} \subset \mathbb{R}^n.$$

level set of f corresponding to c.

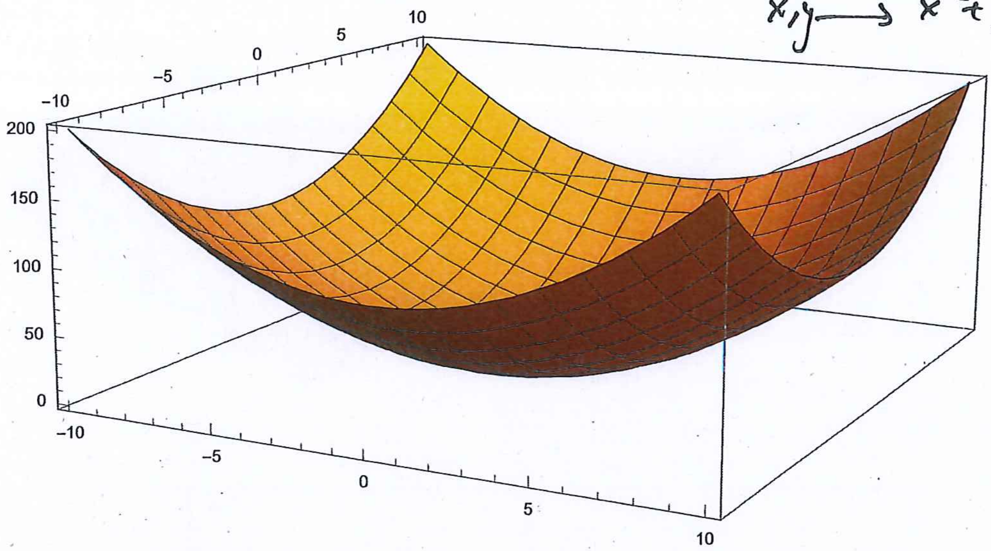
In case of $\mathbb{R}^2$ it is called a level curve.

In $\mathbb{R}^3$, they're called level surfaces.

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$x,y \longrightarrow x^2+y^2$$

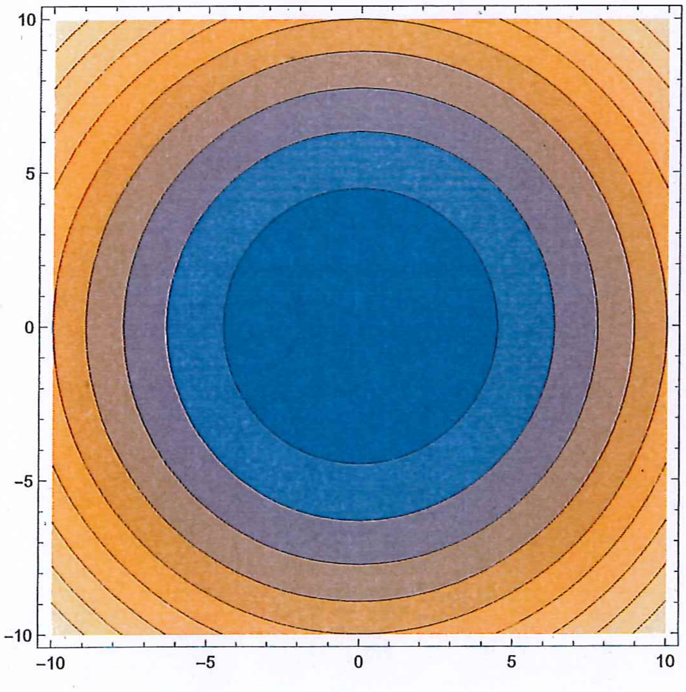
In[2]:= `Plot3D[x^2 + y^2, {x, -10, 10}, {y, -10, 10}]`

Out[2]=



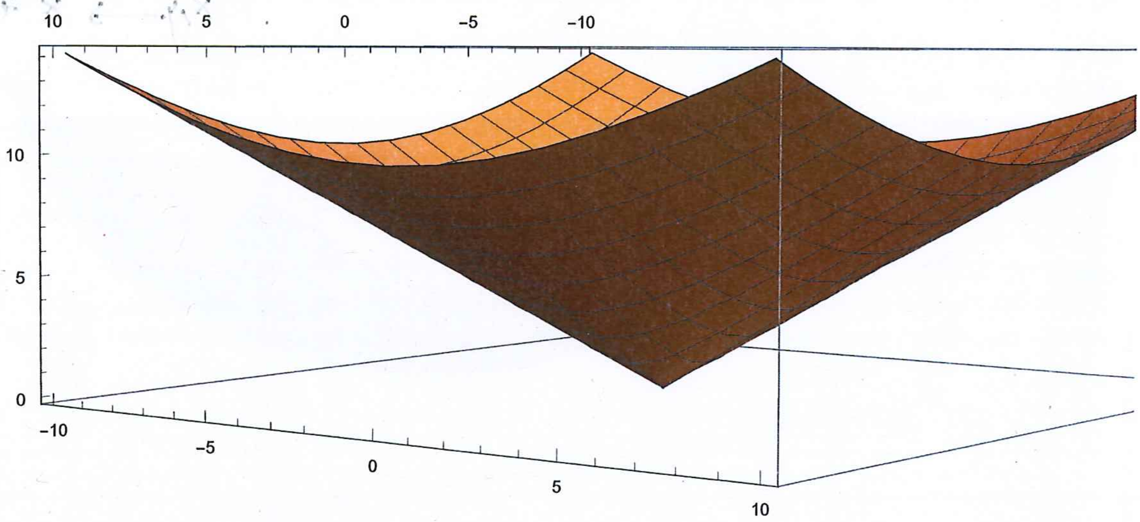
In[20]:= `ContourPlot[ x^2 + y^2 , {x, -10, 10}, {y, -10, 10}]`

Out[20]=



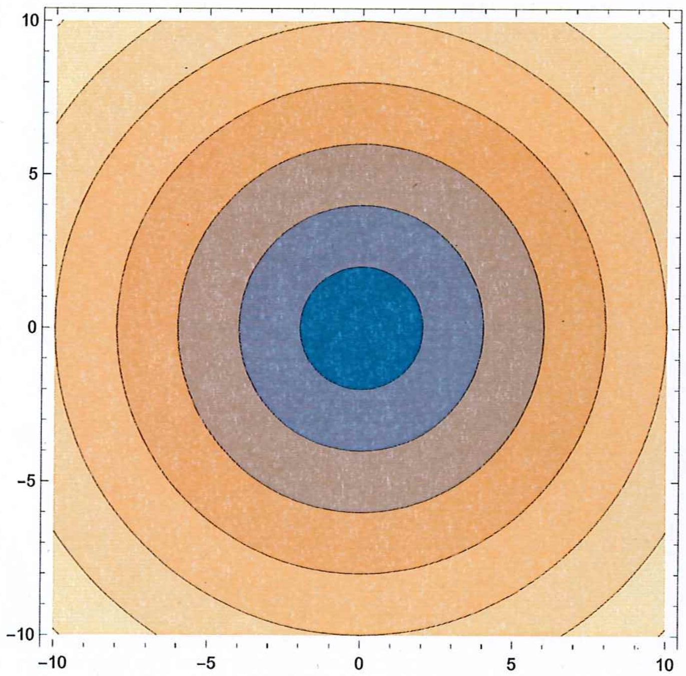
In[10]:= Plot3D[Sqrt[x^2 + y^2], {x, -10, 10}, {y, -10, 10}]

Out[10]=



In[21]:= ContourPlot[Sqrt[x^2 + y^2] , {x, -10, 10}, {y, -10, 10}]

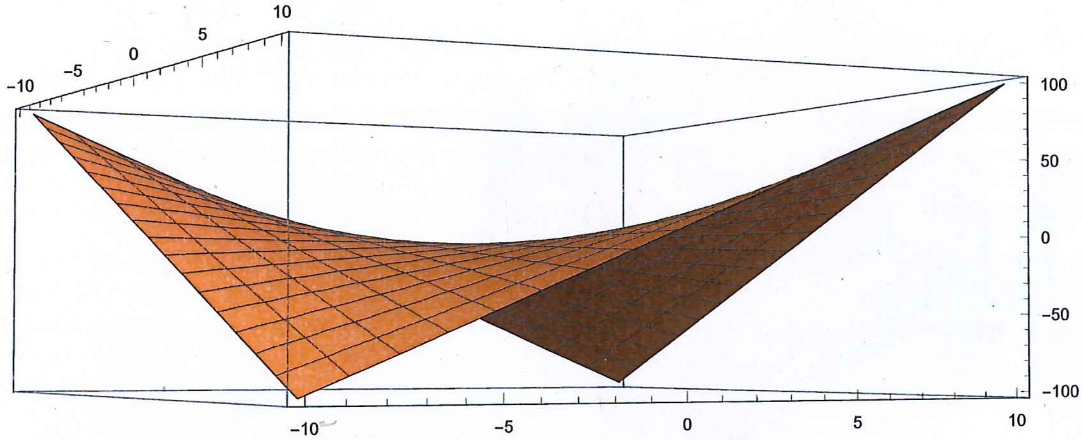
Out[21]=



xy

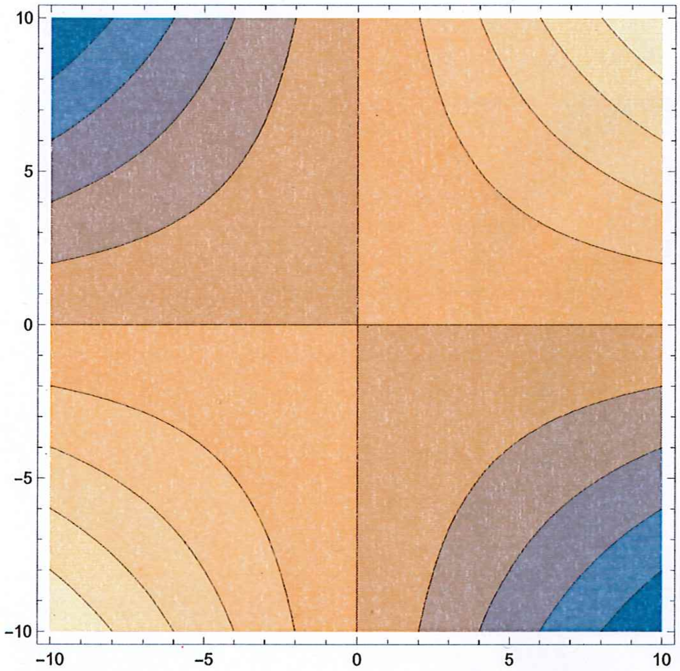
In[37]:= `Show[%36, ImageSize → Large]`

Out[37]=



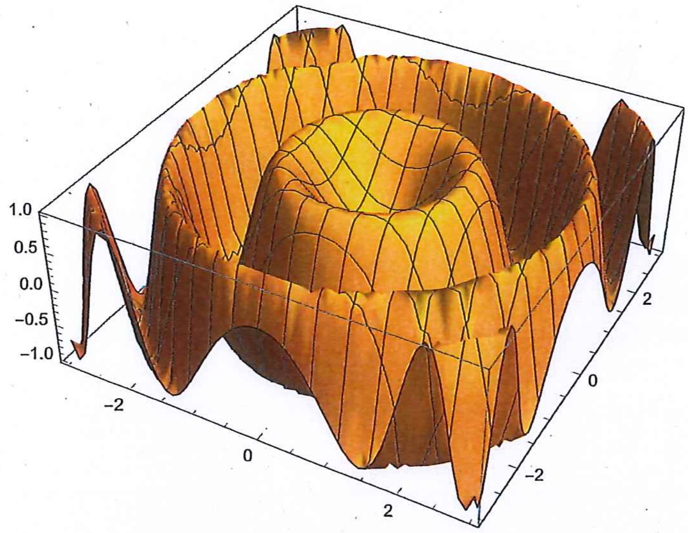
In[38]:= `ContourPlot[x * y, {x, -10, 10}, {y, -10, 10}]`

Out[38]=



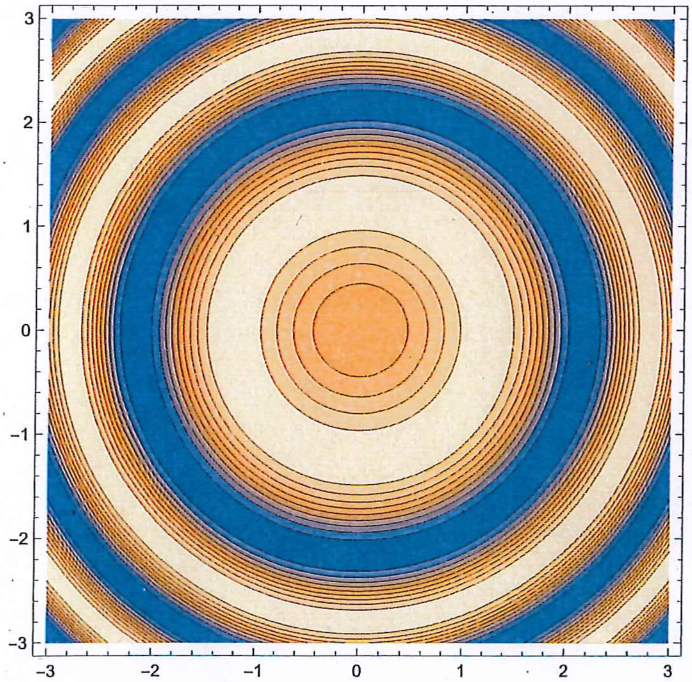
In[27]:= `Plot3D[Sin[x^2 + y^2], {x, -3, 3}, {y, -3, 3}]`

Out[27]=



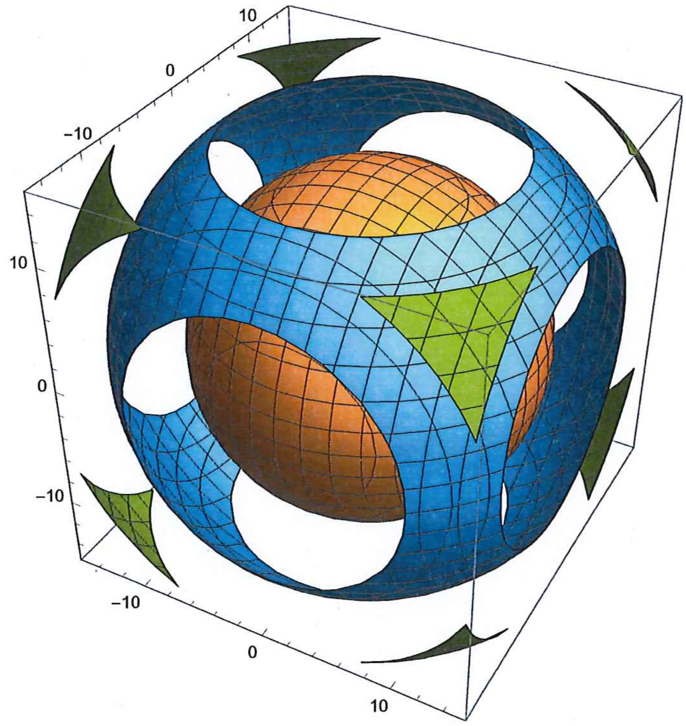
In[26]:= `ContourPlot[Sin[x^2 + y^2], {x, -3, 3}, {y, -3, 3}]`

Out[26]=

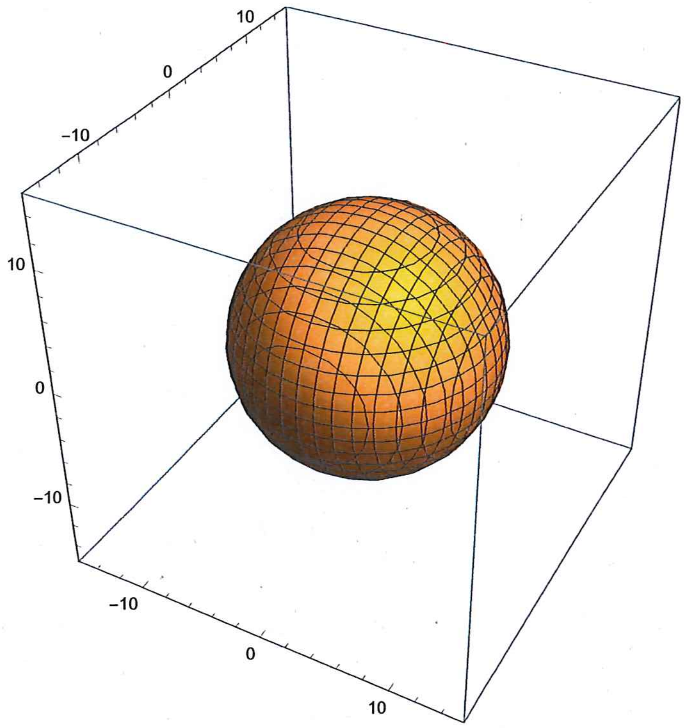


In[36]:= `Plot3D[x * y , {x, -10 , 10 }, {y, -10 , 10 }]`

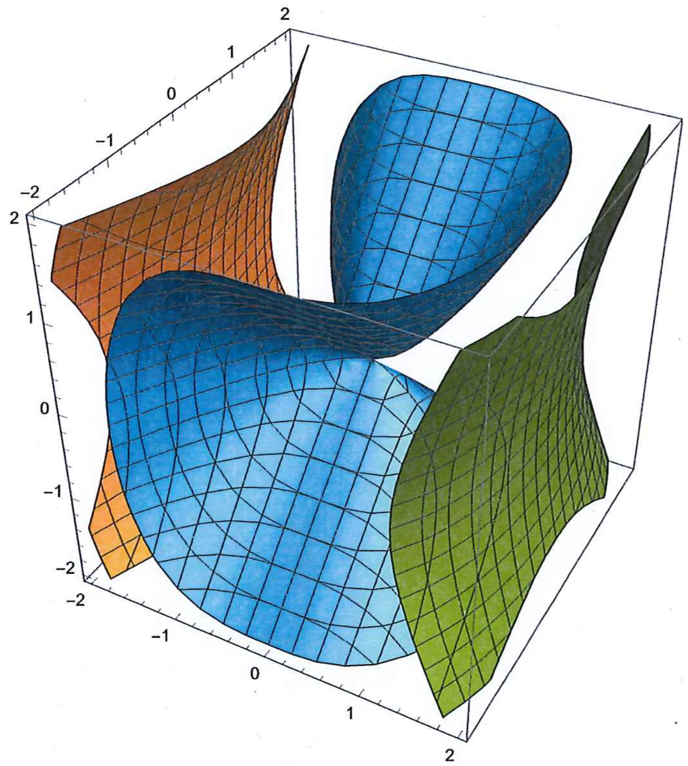
```
ContourPlot3D[x^2 + y^2 + z^2 , {x, -15, 15}, {y, -15, 15}, {z, -15, 15}]
```



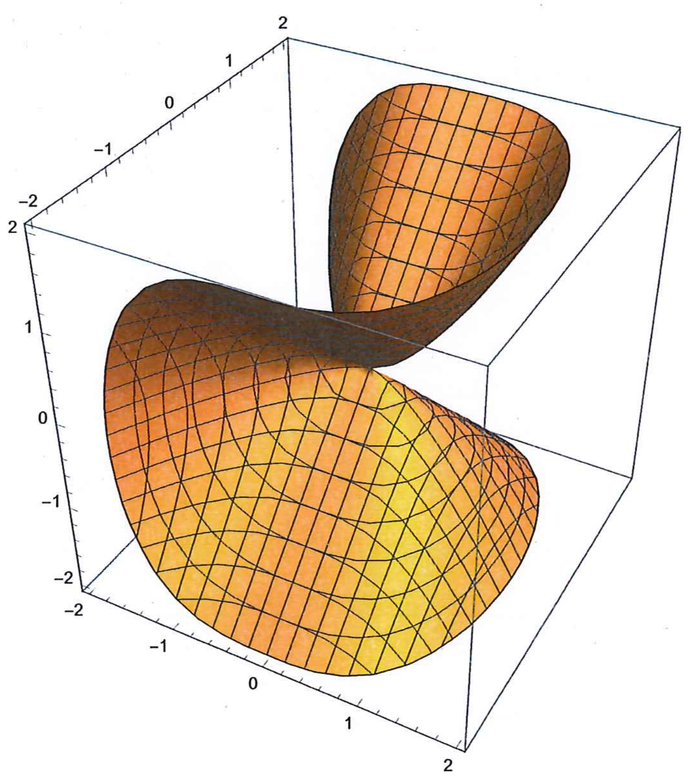

```
ContourPlot3D[x^2 + y^2 + z^2 == 100, {x, -15, 15}, {y, -15, 15}, {z, -15, 15}]
```



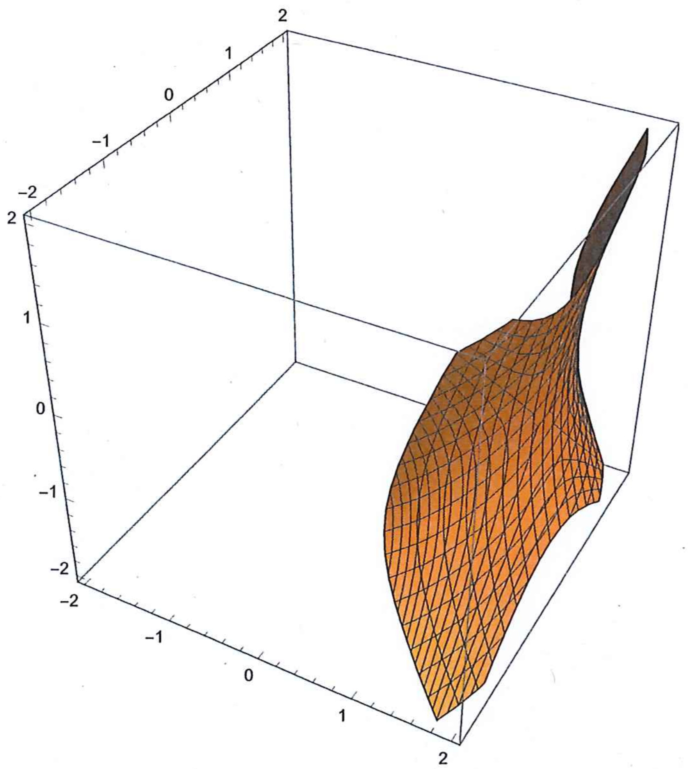
```
ContourPlot3D[x^3 + y^2 - z^2, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```




```
ContourPlot3D[x^3 + y^2 - z^2 == 0, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```



```
ContourPlot3D[x^3 + y^2 - z^2 == 6, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}]
```

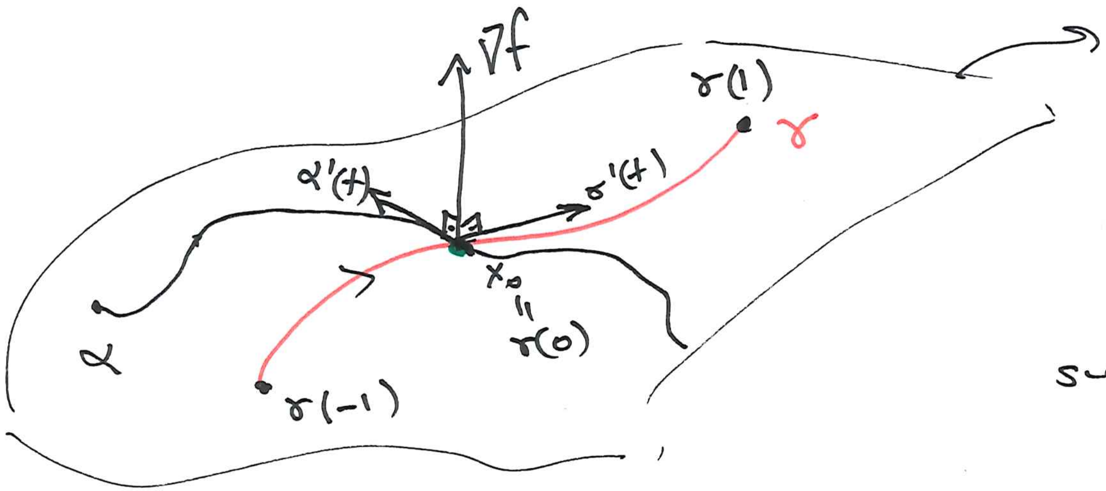


Geometric interpretation of ∇f

$f : \mathbb{R}^n \to \mathbb{R}$ $\quad n \geq 3$, $\quad x_0 \in \mathbb{R}^n$, with $f(x_0) = c$.

Then $x_0 \in L(c) := \{ x \mid f(x) = c \} = f^{-1}(\{c\})$.

if $n=3$ this is a surface.



$f^{-1}(c)$

Let $\gamma : [-1, 1] \to \mathbb{R}^n$ be a diff curve through x_0. such that $\gamma([-1,1]) \subset f^{-1}(c)$.

Then $\quad f(\gamma(t)) = c \qquad t \in [-1, 1]$.

By chain rule $\quad \underbrace{df(\gamma(t))}_{\nabla f(\gamma(t))} \cdot \gamma'(t) = 0$.

$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0$.

In particular $\nabla f(x_0)$, $\gamma'(0)$ are perpendicular.

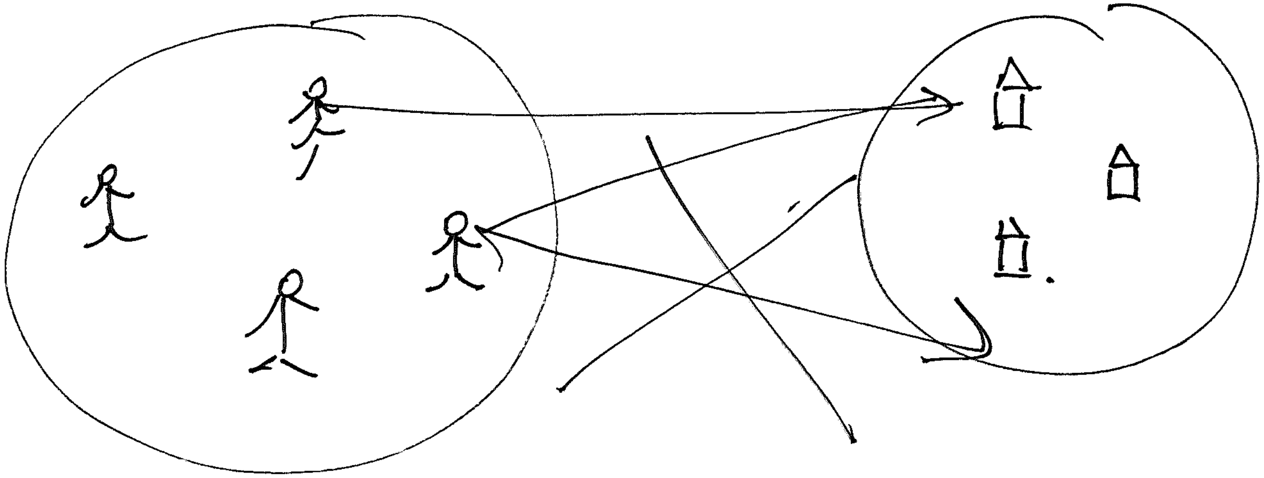
Since this is true for any curve through x_0 that lies on the surface $f^{-1}(c)$, in fact $\nabla f(x_0)$ is perpendicular to the plane ~~gene~~ that is generated by the tangent vectors to the curves in $f^{-1}(c)$ through x_0.

Clicker

if $L(c) \cap L(d) \neq \emptyset \qquad c \neq d$

$\{(x,y) \mid f(x,y) = c\} \cap \{(x,y) \mid f(x,y) = d\}$.

then $\exists\ (x_0, y_0)$ st $c = f(x_0, y_0) = d$,



§ Higher partial derivatives, Hessian, Taylor Formula.

$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

Recall we say $f \in C^1(\Omega)$ if f is differentiable and $\frac{\partial f}{\partial x^i}$ are continuous.

Defn We say f is of class $C^2(\Omega)$ if $\frac{\partial f}{\partial x^i} \in C^1(\Omega)$.

Ex: $f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

The mixed partial $\dfrac{\partial f}{\partial x\,\partial y} \neq \dfrac{\partial f}{\partial y\,\partial x}$.

$f: \mathbb{R}^n \to \mathbb{R}$

Thm Let $f \in C^2(\Omega)$ (ie $\dfrac{\partial f}{\partial x^i}$, $\dfrac{\partial f}{\partial x^i \partial x^j}$ exist and continous.).

Then $\dfrac{\partial^2 f}{\partial x^i \partial x^j} = \dfrac{\partial}{\partial x^i}\left(\dfrac{\partial f}{\partial x^j}\right) = \dfrac{\partial}{\partial x^j}\dfrac{\partial f}{\partial x^i} = \dfrac{\partial^2 f}{\partial x^j \partial x^i}$

$1 \leq i, j \leq n$.

Defn Let $f : \mathbb{R}^n \to \mathbb{R}$. Assume that the second order partial derivatives $\dfrac{\partial f}{\partial x^i \partial x^j}$ of f at x_0 exist. The $n \times n$ matrix of 2nd order partial derivatives

$$\left(D_{ij}(f)(x_0) \right)_{1 \le i, j \le n}$$

is called the Hessian Matrix of f

Ex: $f : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \to x^2 y + yz$

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2 + z \qquad \frac{\partial f}{\partial z} = y$$

$$\frac{\partial^2 f}{\partial x^2} = 2y \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x \qquad \frac{\partial^2 f}{\partial x \partial z} = 0.$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \frac{\partial^2 f}{\partial y \partial z} = 1$$

$$\frac{\partial^2 f}{\partial x \partial z} = 0 \qquad \frac{\partial^2 f}{\partial y \partial z} = 1 \qquad \frac{\partial^2 f}{\partial z^2} = 0.$$

$$\text{Hessian}(f) = \text{Hess}(f) = \nabla^2 f = \begin{pmatrix} 2y & 2x & 0 \\ 2x & 0 & 1 \\ 0 & 1 & 0. \end{pmatrix}$$

Note for $f \in C^2$, mixed partial derivatives $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$

$D_{ij} f = D_{ji} f \implies \text{Hess}(f)$ is a symmetric matrix.

$f : \mathbb{R} \to \mathbb{R}$

$f'(x)$

$f''(x)$

Recall the first order Taylor approximation of $f : \mathbb{R}^n \to \mathbb{R}$ (which is diff in x_0) is given by the gradient, namely.

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + R(|x - x_0|).$$

(We $f = o(g)$ if $\frac{f}{g} \to 0$ as $x \to a$.)

Ex : Give an approximation for $\alpha = \sqrt{(3.03)^2 + (3.95)^2}$

Soln : $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longrightarrow \sqrt{x^2+y^2}$ $\qquad (x_0, y_0) = (3, 4)$

$$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$f(3.03, 3.95) \approx f(3,4) + \nabla f(3,4) \cdot (3.03-3, 3.95-4)$$

$$= 5 + \left(\frac{3}{5}, \frac{4}{5}\right) \cdot (0.03, -0.05).$$

$$= 5 + \frac{3}{5} \cdot (0.03) - \frac{4}{5} \cdot (0.05) \sim 4.978.$$

actual value $\sim 4.97829 \ldots$