$$\left.\begin{array}{l} f: \mathbb{R} \longrightarrow \mathbb{R}. \\ f: \mathbb{R} \longrightarrow \mathbb{R}^n. \\ \quad x \longrightarrow (f_1(x), \ldots f_n(x)) \\ f_i(x): \mathbb{R} \longrightarrow \mathbb{R}. \end{array}\right\} \text{Analysis I.}$$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad$ ($n=1$ $\rightarrow$ Analysis 1.)

$\quad \bar{x} \longrightarrow (f_1(\bar{x}), \ldots f_m(\bar{x}))$

$f_i: \mathbb{R}^n \longrightarrow \mathbb{R}.$ $\quad (m=1)$

When $m=1$, $f: \mathbb{R}^n \to \mathbb{R}$ is sometimes called a scalar field.

When $m>1$ $f: \mathbb{R}^n \to \mathbb{R}^m$ is called a vector field.

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad\quad$ $f: \mathbb{R}^2 \to \mathbb{R}.$

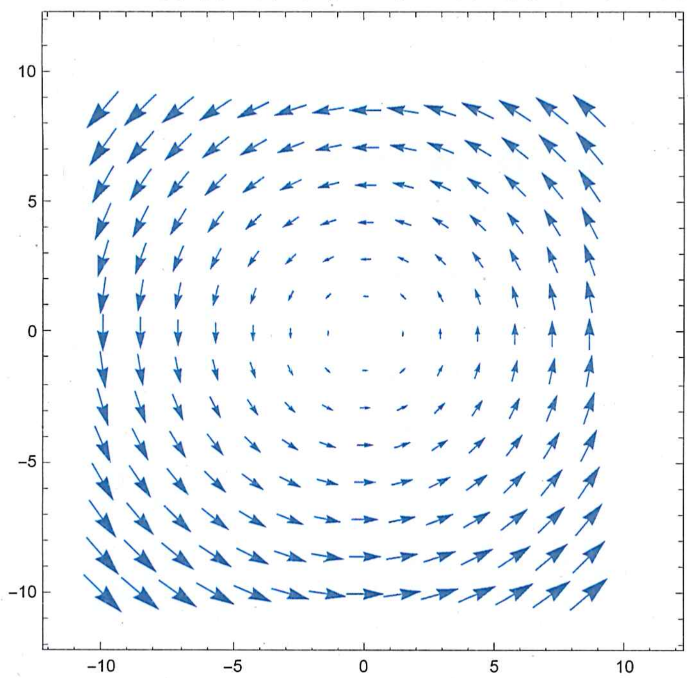
$3\frac{1}{2}$

In[5]:= `VectorPlot[{-y, x}, {x, -10, 10}, {y, -10 , 10 }]`

$f = \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \to (-y, x)$

Out[5]=

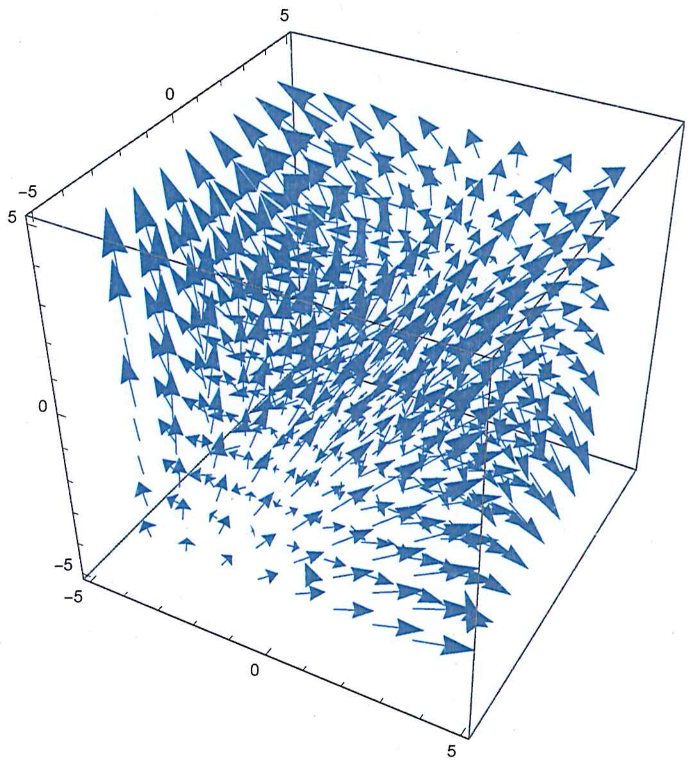


In[2]:= `VectorPlot3D[{x, -y, z + 1}, {x, -4, 4}, {y, -4, 4}, {z, -4, 4}]`

$f = \mathbb{R}^3 \to \mathbb{R}^3$

$(x, y, z) \to (x, -y, z+1)$.

Out[2]=



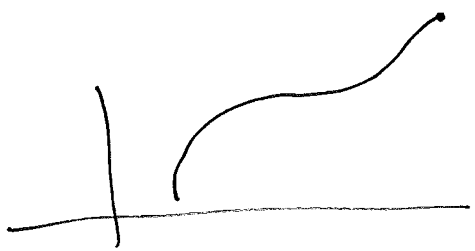
$f: \mathbb{R}^n \to \mathbb{R}^m.$

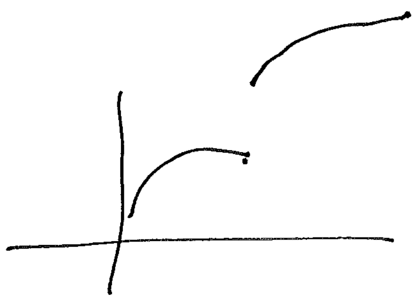
$x \to (f_1(x), \ldots f_m(x))$

$f_i(x): \mathbb{R}^n \to \mathbb{R}.$

We'll start with functions $f: \mathbb{R}^n \to \mathbb{R}$

ie. scalar field.

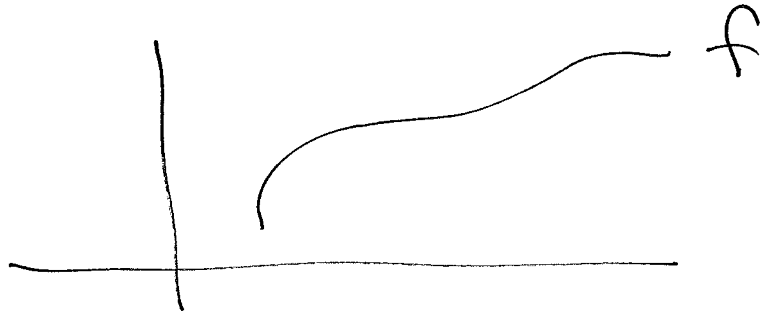




$f: \mathbb{R}^2 \to \mathbb{R}$

$\text{Grapf}(f) = \{(x, y, z) \mid f(x,y) = z\} \subset \mathbb{R}^3$

$f: \mathbb{R} \to \mathbb{R}$ $\quad$ $\text{grapf}(f) = \{(x,y) \mid f(x) = y\} \subset \mathbb{R}^2$



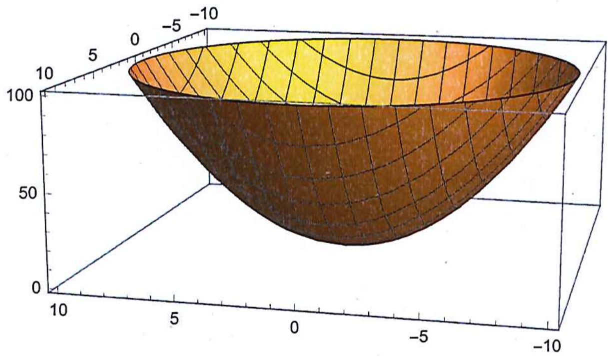
$4\frac{1}{2}$

In[9]:=
```
Plot3D[x^2 + y^2, {x, -10, 10}, {y, -10, 10},
  RegionFunction → Function[{x, y, z}, x^2 + y^2 < 100]]
```

$f = \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \Longrightarrow x^2+y^2$

Out[9]=

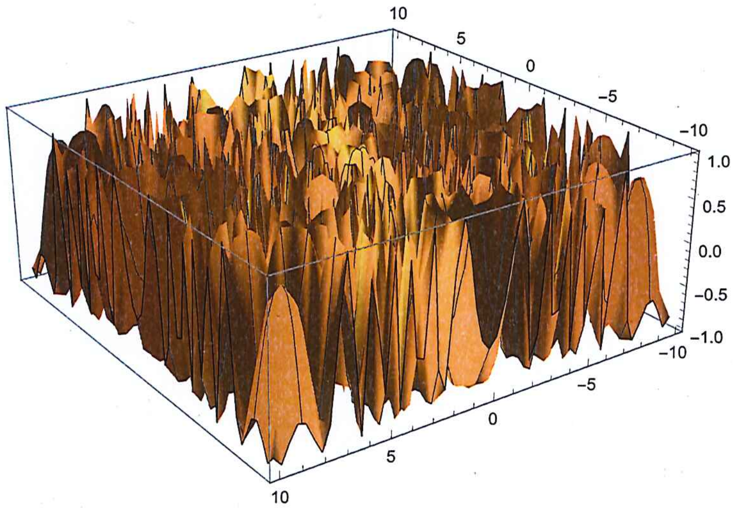


In[10]:=
```
Plot3D[Sin[x^2 + y^2], {x, -10, 10}, {y, -10, 10}]
```

$f = \mathbb{R}^2 \to \mathbb{R}$

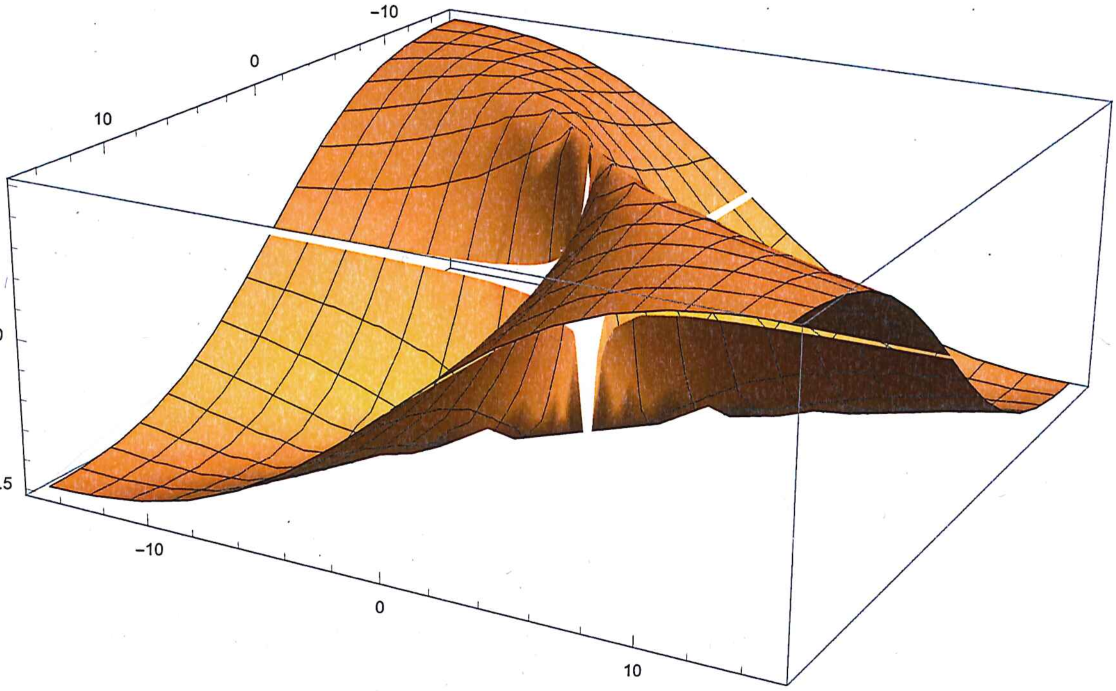
$x,y \to \sin(x^2+y^2).$

Out[10]=



```
In[10]:= Plot3D[x * y / (x^2 + y^2), {x, -15, 15}, {y, -15, 15}, Exclusions → {x == 0, y == 0}]
```

Out[10]=



$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x,y) \mapsto \begin{cases} xy/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Geogebra.

Limits and continuity (stetigkeit).

<u>Defn.</u> $f: \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

We say b is the limit of f as x tends to a, and write $\lim_{x \to a} f(x) = b$

if
$$\lim_{\|x-a\| \to 0} \|f(x) - b\| = 0 \quad (*)$$

$(*)$ is the classical limit from analysis I.

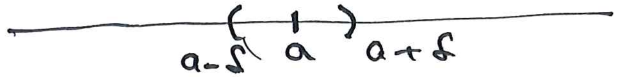
__Recall.__ $f: \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$

we say $\lim_{x \to a} f(x) = b$ if

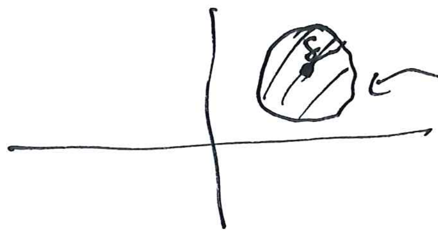
$\forall \varepsilon > 0$, $\exists \delta > 0$ so that $\forall x$ with $|x - a| < \delta$ we have $|f(x) - b| < \varepsilon$.

__Defn.__ $f: \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. we say $\lim_{x \to a} f(x) = b$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ so that $\forall x \in \mathbb{R}^n$ with $\|x - a\| < \delta$ we have $\|f(x) - b\| < \varepsilon$.



$\mathbb{R}$.



$\mathbb{R}^2$ $\|x-a\| < \delta$.

Similarly we can define continuity.

Defn: A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\underline{a} \in \mathbb{R}^n$ if f is defined at a and if $\lim_{x \to a} f(x) = f(a)$.

Basic results about limits of sums and "products" of functions extend to $\mathbb{R}^n$.

Thm 1. Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^n$, $b, c \in \mathbb{R}^m$

If $\lim_{x \to a} f(x) = b$, and $\lim_{x \to a} g(x) = c$ then we have

1) $\lim_{x \to a} f(x) + g(x) = b + c$

2) $\lim_{x \to a} \lambda f(x) = \lambda b \quad \forall \lambda \in \mathbb{R}$.

3) $\lim_{x \to a} f(x) \cdot g(x) = b \cdot c$ $\qquad b \cdot c = \langle b, c \rangle$ the standard scalar product.

4) $\lim_{x \to a} \|f(x)\| = \|b\|$

Examples. 1) The identity function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longrightarrow x$$

is continous $\forall x \in \mathbb{R}^n$.

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longrightarrow (f_1(x), \ldots f_n(x))$$

$$f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x = (x_1, \ldots x_n) \longrightarrow x_i$$

2) Linear Transformations

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear

Then T is continous at $a \in \mathbb{R}^n$ $\forall a \in \mathbb{R}^n$.

T linear : $T(\underset{\cdot}{a} + \underset{\cdot}{h}) = T(a) + T(\underset{\cdot}{h})$

w.t.s $\lim_{x \to a} T(x) = T(a)$

Hence by linearity it is enough to show that

$T(h) \to 0$ as $h \to 0$.

Write $h = h_1 e_1 + h_2 e_2 + \dots + h_n e_n$ $\quad e_i = (0, \dots 0, 1, 0 \dots 0)$ standard basis.

$$T(h) = T(h_1 e_1) + \dots + T(h_n e_n)$$
$$= h_1 T(e_1) + h_2 T(e_2) + \dots + h_n T(e_n) \quad \text{by linearity.}$$

$h \to 0 \iff h_i \to 0 \quad \forall\, i = 1, \dots, n.$

Hence as $h \to 0$ $\quad T(h) \to 0$.

3) Polynomials in n-variables are continous.

$P : \mathbb{R}^n \to \mathbb{R}$

$$P(x) = \sum_{k_1=0}^{r_1} \dots \sum_{k_n=0}^{r_n} c_{k_1 \dots k_n} x_1^{k_1} \dots x_n^{k_n}.$$

4) Rational functions $f : \mathbb{R}^n \to \mathbb{R}$

$f(x) = P(x)/Q(x)$ $\quad P, Q$ polynomials

are continous $\forall\, x \in \mathbb{R}^n$ for which $Q(x) \neq 0$.

Pf 3) w.t.s. $\|f(x)\cdot g(x) - b\cdot c\| \to 0$ as $\|x-a\| \to 0$.

$$f(x)\cdot g(x) - b\cdot c = (f(x)-b)\cdot(g(x)-c) + b\cdot(g(x)-c) + c\cdot(f(x)-b)$$

$$0 \le \|f(x)\cdot g(x) - b\cdot c\| \le \|(f(x)-b)\cdot(g(x)-c)\| + \|b\cdot(g(x)-c)\| + \|c\cdot(f(x)-b)\|.$$

Δ-inequality.

($\|u\cdot v\| \le \|u\|\,\|v\| \to$ Cauchy Schwarz.).

$$\le \|f(x)-b\|\,\|g(x)-c\| + \|b\|\,\|g(x)-c\| + \|c\|\,\|f(x)-b\|$$

CS.

Since $\lim f(x) = b \Rightarrow \|f(x)-b\| \xrightarrow[\|x-a\|\to 0]{} 0$
$\lim g(x) = c \Rightarrow \|g(x)-c\| \xrightarrow[\|x-a\|\to 0]{} 0$.
$\Longrightarrow \|f(x)\cdot g(x) - b\cdot c\| \to 0$ as wanted.

4) This follows from 3) by taking $g = f$.

(15).

Thm 2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$
$x \to (f_1(x) \ldots f_m(x))$

with $f_i : \mathbb{R}^n \to \mathbb{R}$

Then f is continoust at $x = a$
if and only if f_i's are cont. at $x = a$, $\forall i$.

Thm 3. Let f, g be 2 functions, such that
$f \circ g$ is defined at a point a, where
$(f \circ g)(x) = f(g(x))$.

If g is continuous at a, and f is continous at $g(a)$ then $f \circ g$ is continous at $x = a$.

<u>Ex</u> 1) $\cos(x^2y)$ cont. $\forall (x,y) \in \mathbb{R}^2$.

2) $\log(x^2+y^2)$ is cont. $\forall (x,y) \in \mathbb{R}^2 \setminus \{0,0\}$.

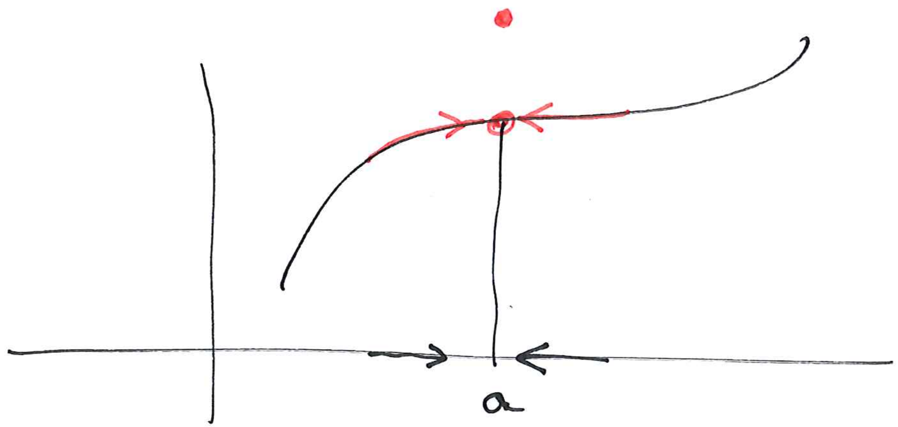
3) $\log(\cos(x^2+y^2))$ cont. $\forall (x,y) \in \mathbb{R}^2 \setminus U$

$$U = \{(x,y) \mid (x^2+y^2) = \left(\frac{2n+1}{2}\right)\pi \text{ for some } n\}$$

<u>Remark</u>: In dimension 1, the discontinuity of functions are at points..

The discontinuity of functions of 2 variables can be points. as in 2) or it can a collection of curves as in 3)

$$f(x,y) = \begin{cases} xy/x^2+y^2 & (x,y) \neq (0,0) \\ 0. & (x,y) = (0,0) \end{cases} \quad \Bigg\} \text{ Does not have a limit as } (x,y) \to (0,0).$$



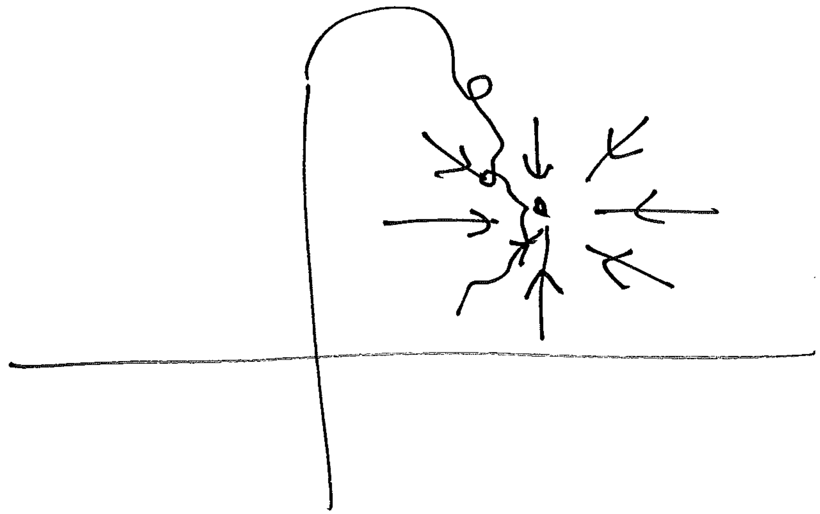
$\lim_{x \to a} f(x)$ exists

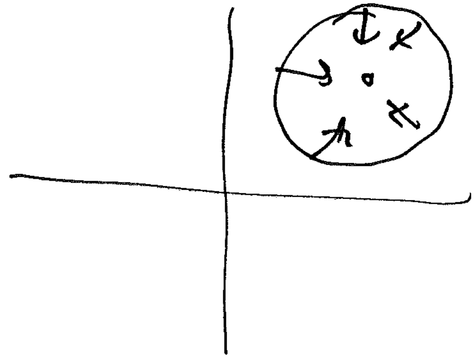
$\Longleftrightarrow$ $\lim_{x \to a^+} f(x) = L^+$ and

$\lim_{x \to a^-} f(x) = L^-$ exists

<u>and</u> $L^+ = L^-$

$\mathbb{R}^2$



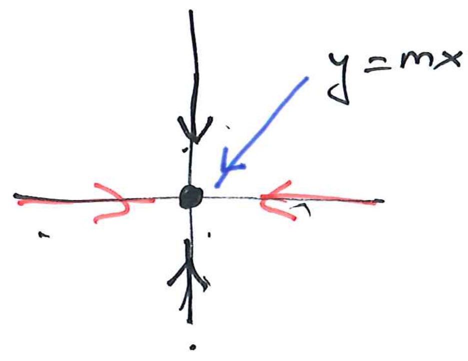


In $\mathbb{R}^2$ there are ∞ly many ways to approach

a pt $\underline{\underline{a}}$

We can look at

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along y axis $(x=0)$



$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{0\cdot y}{0^2+y^2} = 0.$$

Along x-axis

$$\lim f(x,y) = \lim \frac{x\cdot 0}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x(mx)}{x^2+(mx)^2} = \lim_{x\to 0} \frac{m\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \frac{m}{1+m^2}$$

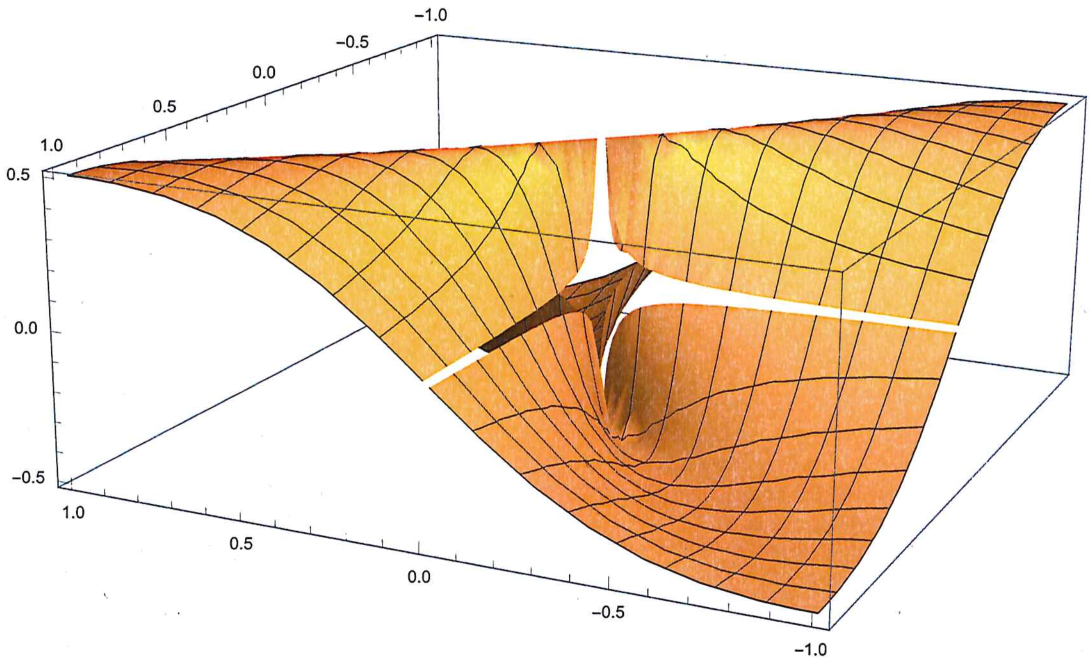
$\neq$

Along $y=mx$

Thm If $\lim\limits_{x\to a} f(x)$ exists then it is unique!

In[4]:= `Plot3D[x * y / (x^2 + y^2), {x, -1, 1}, {y, -1, 1}, Exclusions → {x == 0, y == 0}]`

Out[4]=



$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For studying limits of scaler fields the following lemma can be useful.

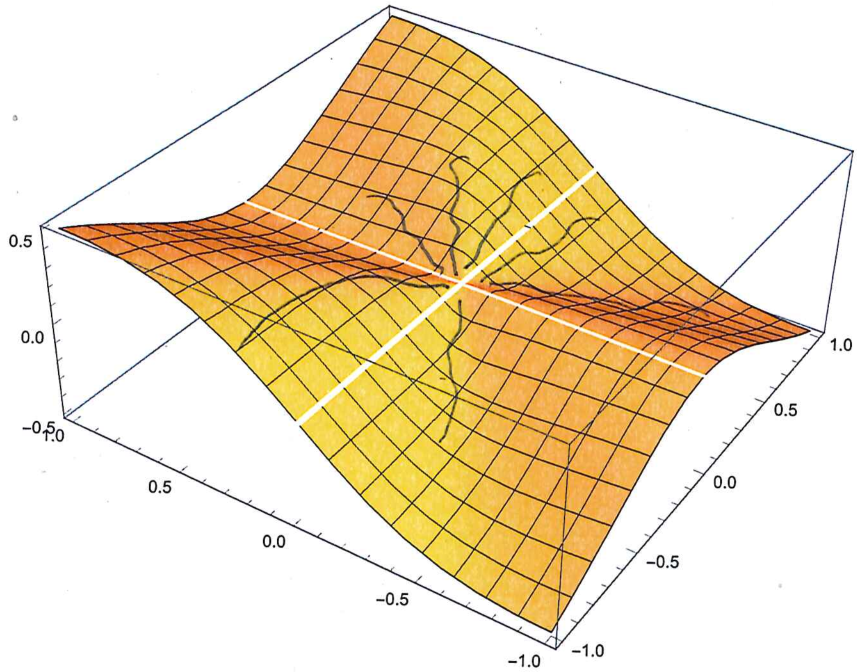
<u>Lemma</u> (<u>Sandwich lemma</u>). Let f, g, h be functions $\mathbb{R}^n \to \mathbb{R}$ where

$f(x) \leq g(x) \leq h(x)$. $\forall x$, let $a \in \mathbb{R}^n$.

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$ then $\lim g$ exists and is equal to L.

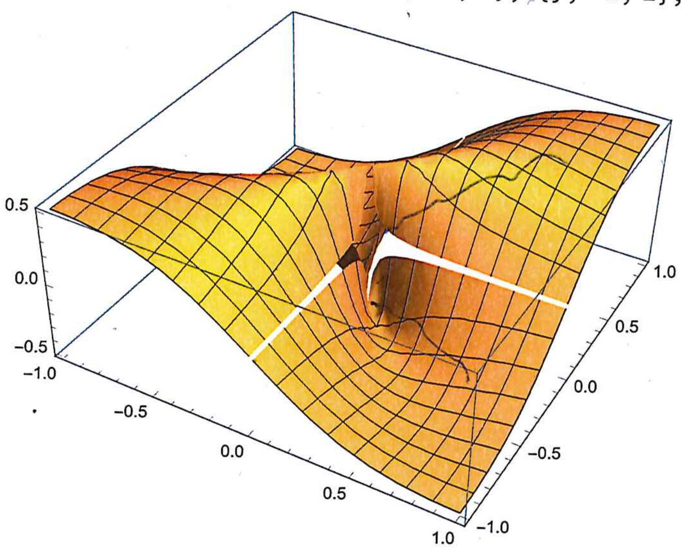
In[2]:= `Plot3D[x^2 * y / (x^2 + y^2), {x, -1, 1}, {y, -1, 1}, Exclusions → {x == 0, y == 0}]`

Out[2]=



In[3]:= `Plot3D[x * y / (x^2 + y^2), {x, -1, 1}, {y, -1, 1}, Exclusions → {x == 0, y == 0}]`

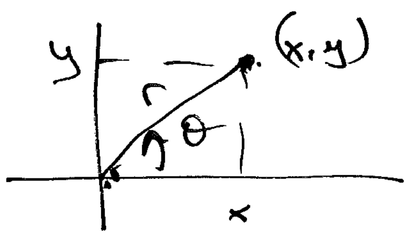
Out[3]=



$$\underset{\downarrow \atop 0}{0} < \underbrace{\|f(x,y)\|} = \|y\| \cancel{\left\|\frac{x^2}{x^2+y^2}\right\|} \underbrace{\left\| \frac{x^2}{x^2+y^2} \right\|}_{\leq 1} \leq \|y\| = |y| \xrightarrow[(x,y)\to 0]{} 0$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) \Rightarrow 0$.

<u>Remark</u>. For studying limits in $\mathbb{R}^2$, it is sometimes helpful to use polar coordinates instead.



$x = r\cos\theta$ $\qquad r = \sqrt{x^2+y^2}$.

$y = r\sin\theta$

For $f = \begin{cases} x^2y/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$f(x,y) = f(r,\theta) = \frac{(r\cos\theta)^2(r\sin\theta)}{r^2} = r\cos^2\theta\sin\theta \to 0 \text{ as } r \to 0.$$

$(x,y) \to (0,0) \iff r \to 0.$

$f = \begin{cases} xy/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$ $\quad f(r,\theta) = \frac{(r\cos\theta)(r\sin\theta)}{r^2}$

$= \cos\theta\sin\theta.$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = \cos\theta\sin\theta.$ Hence

$\lim_{r\to 0} f(r,\theta)$ does not exist. It depends on θ!