Change of variables in multiple Integrals.

Thm let $U \subset \mathbb{R}^n$ be open,

$\varphi : U \longrightarrow \mathbb{R}^n$ an injective C^1 map

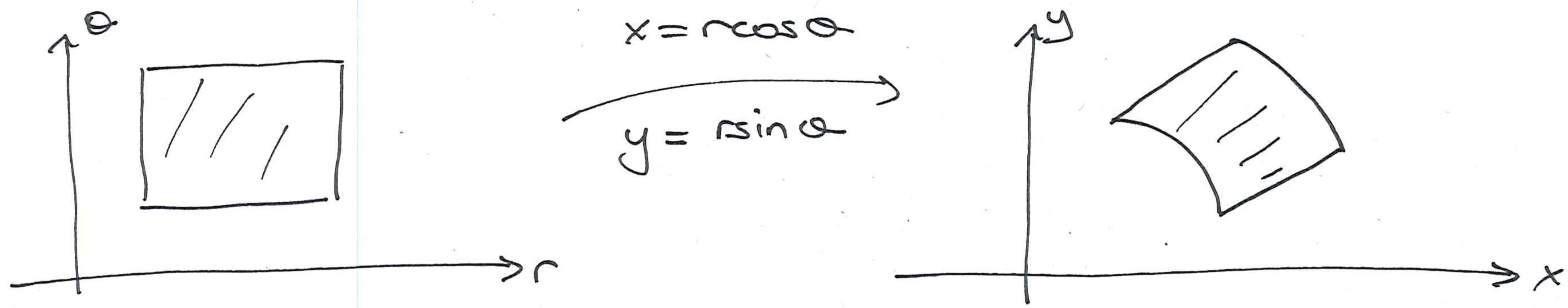
with $\det(\nabla\varphi)(u) \neq 0 \quad \forall u \in U$

where ∇U is the Jacobian Matrix. let $X = \varphi(U)$

and $f : X \longrightarrow \mathbb{R}$ a continuous scalar field. Then

$$\int_{X = \varphi(U)} f(x)\, d\mu(x) = \int_U f(\varphi(u))\, \left|\det(\nabla\varphi(u))\right|\, d\mu(u).$$

Eg: Polar coordinates.



$$\varphi\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

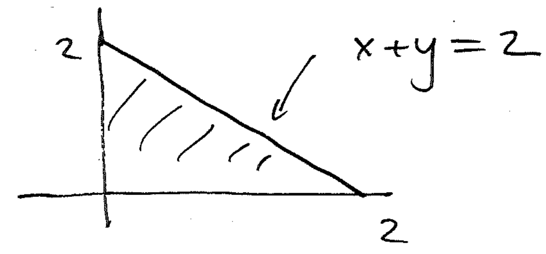
$$\nabla\varphi = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$|\det(\nabla\varphi)| = r$$

$$\iint_{X = \varphi(U)} f(x,y)\,dx\,dy = \iint_{U} f(\varphi(r,\theta))\, r\,dr\,d\theta$$

Another special type of change of variables are linear transformations.

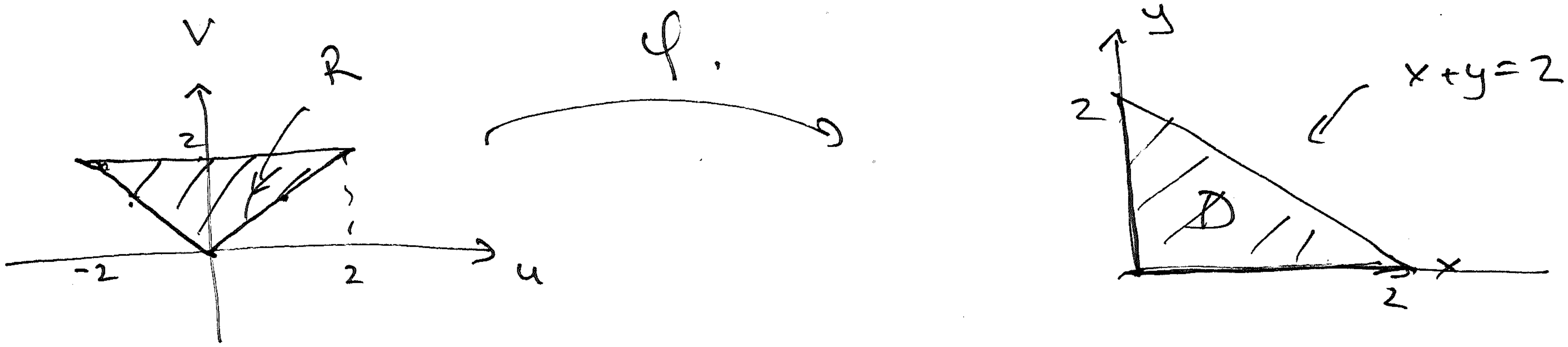
<u>Ex.</u>: $$\int_D e^{(y-x)/y+x} \, dx \, dy$$



We set $\begin{aligned} u &= y - x \\ v &= y + x \end{aligned}$ $\qquad \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \qquad \begin{aligned} x &= \frac{1}{2}(v-u) \\ y &= \frac{1}{2}(v+u). \end{aligned}$$

$$d\varphi = \nabla \varphi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \Rightarrow |\det(\nabla \varphi)| = \frac{1}{2}.$$



$x=0$ line $\Rightarrow$ $\left.\begin{array}{l} u = y - x \Rightarrow u = y \\ v = y + x \Rightarrow v = y \end{array}\right\} \Rightarrow u = v.$

$y=0$ line $\Rightarrow$ $\left.\begin{array}{l} u = -x \\ v = x \end{array}\right\} \Rightarrow u = -v.$

$x+y=2$ $\Rightarrow$ $u = y - (2-y)$

$v = y + (2-y) = 2.$

$$\int_{D=\varphi(R)} e^{(y-x)/y+x} \, dx\, dy = \int_R e^{u/v} \frac{1}{2} \, du\, dv.$$

$$= \frac{1}{2} \int_0^2 \int_{-v}^{v} e^{u/v} \, du\, dv.$$

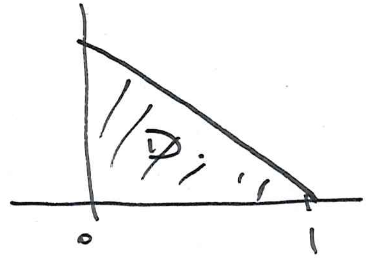
(4).

$$= \frac{1}{2}\int_0^2 \left(v e^{u/v} \Big|_{u=-v}^{u=v} \right) dv. = \frac{1}{2}\int_0^2 (ve - ve^{-1})\, dv.$$

$$= \frac{1}{2}(e-e^{-1}) \int_0^2 v\, dv. = \frac{1}{2}(e-e^{-1})\left(\frac{v^2}{2}\Big|_0^2\right)$$

$$= e - e^{-1}.$$

Ex $$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dy\, dx$$

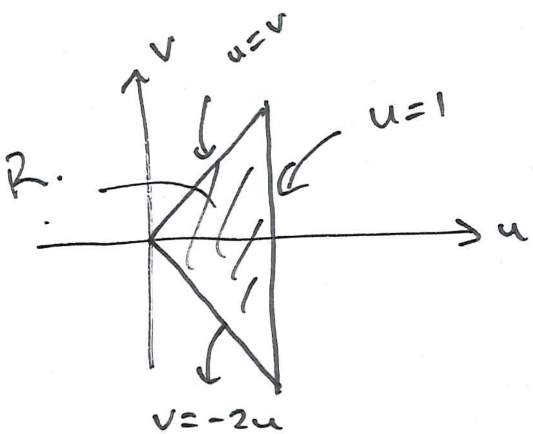


let $u = x+y$
$v = y-2x$

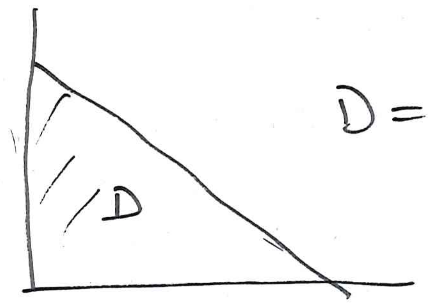
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}}\begin{pmatrix} u \\ v \end{pmatrix}$$

$|\det \varphi| = \frac{1}{3}$



$\xrightarrow{\varphi.}$



$D = \varphi(R).$

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$$x=0 \Rightarrow \left.\begin{array}{l} u=y \\ v=y \end{array}\right\} \Rightarrow u=v.$$

$$y=0 \Rightarrow \left.\begin{array}{l} u=x \\ v=-2x \end{array}\right\} \Rightarrow v=-2u.$$

$$y=1-x \Rightarrow \begin{array}{l} u=x+1-x=1. \Rightarrow u=1 \\ v= \ldots \end{array}$$

$$\int_{D=\varphi(R)} \sqrt{x+y}\,(y-2x)^2\,dy\,dx = \int_R u^{1/2} v^2 \frac{1}{3}\,du\,dv.$$

$$= \int_0^1 \int_{v=-2u}^{v=u} u^{1/2} v^2 \frac{1}{3}\,du\,dv = \ldots \frac{2}{9}.$$

let $X \subset \mathbb{R}^n$ non-compact subset.

$f : X \longrightarrow \mathbb{R}$ such that

$\int_K f \, d\mu(x)$ exists $\forall\, K$. compact sset of X.

__Defn.__ We say that $\int_K f \, d\mu$ for growing K has limit I, if $\forall\, \varepsilon > 0$, $\exists\, K_\varepsilon \subset X$ compact set such that $\forall\, K \subset X$ containing K_ε $(K_\varepsilon \subset K)$

$$\left| \int_K f \, d\mu - I \right| < \varepsilon .$$

Thm. If f is continuous then the improper integral $\int_K f \, d\mu$ exists if and only if

$$\int_{K_0} |f| \, d\mu \text{ is bounded from above } \forall K_0 \text{ compact.}$$

Remark We write $\int_X f \, d\mu = +\infty$ or $-\infty$

if $\int_K f \, d\mu$ has limit $\pm\infty$.

Fact: If $f \geq 0$ or $f \leq 0$ $\forall x$

then $\int_X f \, d\mu$ exists or $\pm \infty$.

and all the previous rules, including Fubini's theorem holds for improper integrals.

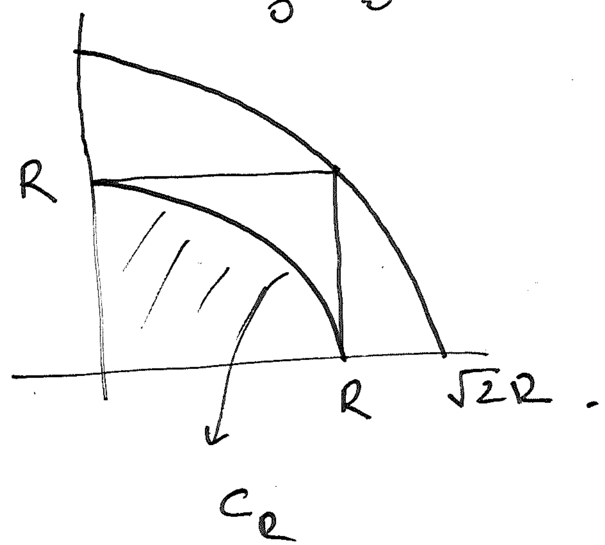
Ex $I = \int_0^\infty e^{-x^2} dx$

We can evaluate this integral by first making it "more complicated"

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy. = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy.$$

$$= \lim_{R\to\infty} \int_0^R e^{-x^2} dx \ \lim_{R\to\infty} \int_0^R e^{-y^2} dy.$$

$$I_R^2 = \int_0^R \int_0^R e^{-(x^2+y^2)} dx\, dy.$$



$$\int_{C_R} e^{-(x^2+y^2)} dx\,dy \le \int_0^R \int_0^R e^{-(x^2+y^2)} dx\,dy$$

$$\le \int_{C_{\sqrt{2}R}} e^{-(x^2+y^2)} dx\,dy$$

$$\int_{C_R} e^{-(x^2+y^2)} dx\,dy = \int_0^{\pi/2} \int_0^R \underbrace{e^{-r^2} r}_{\left(\frac{-1}{2} e^{-r^2}\right)'} dr\, d\theta. = \frac{\pi}{4}\left(1 - e^{-R^2}\right).$$

$$\frac{\pi}{4}\left(1-e^{-R^2}\right) \leq I_R^2 \leq \frac{\pi}{4}\left(1-e^{-2R^2}\right).$$

let $R \to \infty$.

$$\frac{\pi}{4} \leq I^2 \leq \frac{\pi}{4}$$

$$\Rightarrow I^2 = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}.$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

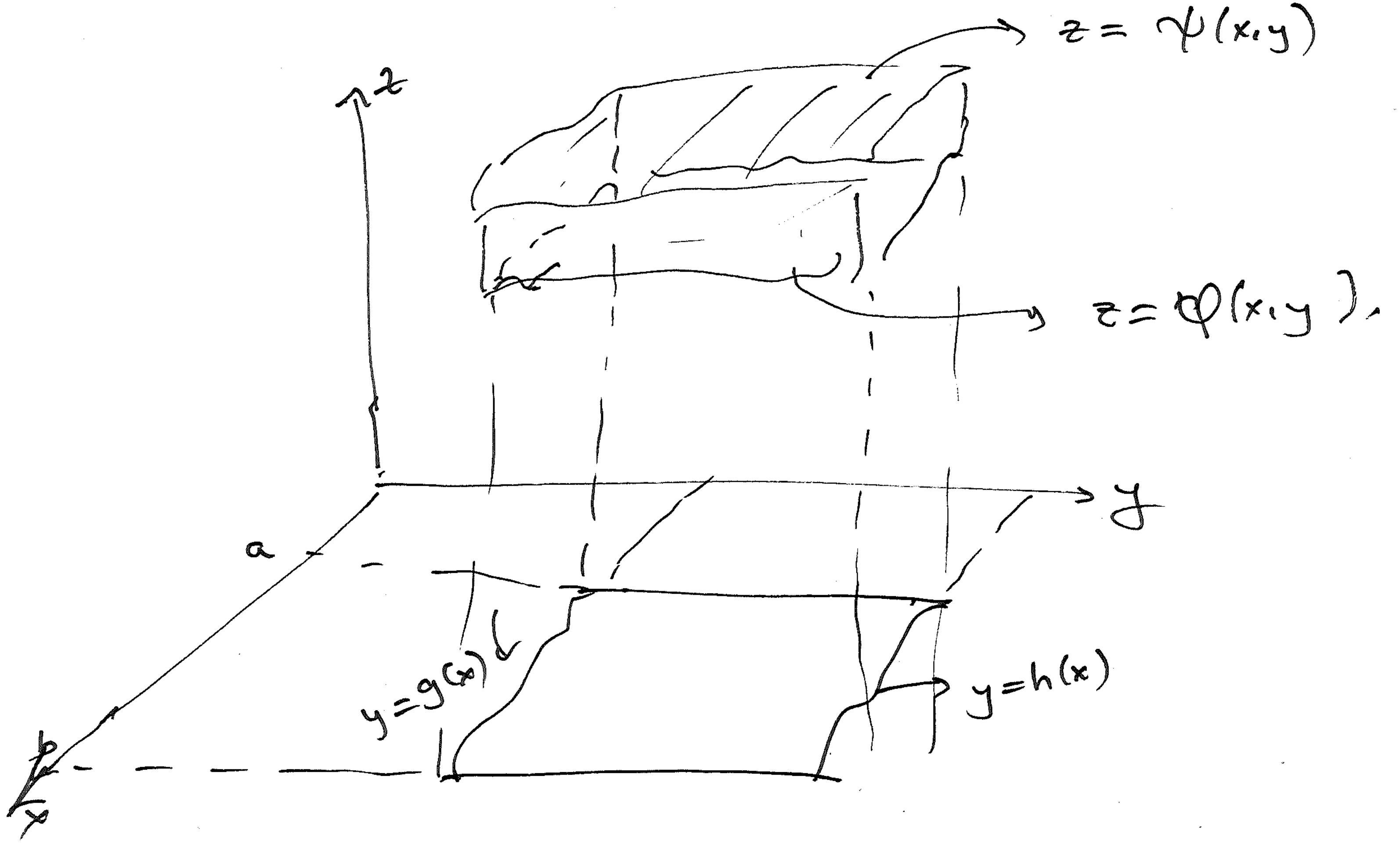
All the theory goes similarly for triple, multiple integrals.

Defn: A subset $D \subset \mathbb{R}^3$ is called a normal region wrt x (resp y, z), if it is of the form

$$D = \{ (x,y,z) \in \mathbb{R}^3 \mid a \leq x \leq b \quad g(x) < y < h(x) \quad \varphi(x,y) < z < \psi(x,y) \}.$$

Thm (Fubini). Let $D \subset \mathbb{R}^3$ be a normal region with respect to x with a representation as above

$$\int_D f \, d\mu = \int_a^b \int_{g(x)}^{h(x)} \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z) \, dz \, dy \, dx$$

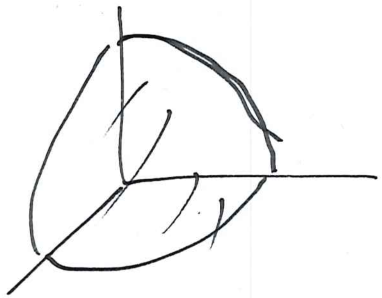


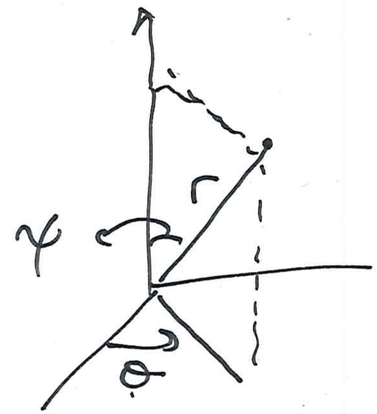
Rk. The integral $\displaystyle\int_D 1 \, d\mu$, $D \subset \mathbb{R}^3$

gives the volume of D.

$$\int_a^b \int_g^h \int_\varphi^\psi 1 \, dz \, dy \, dx = \int_c^b \int_{g(x)}^{h(x)} \left(\psi(x,y) - \varphi(x,y)\right) dy \, dx$$

Ex $\iiint_D 1 \, dv.$ D octant of a unit sphere





$$\varphi(r,\theta,\psi) = \begin{pmatrix} r\cos\theta \sin\psi \\ r\sin\theta \sin\psi \\ r\cos\psi \end{pmatrix}$$

$$|\det(d\varphi)| = r^2 \sin\psi.$$

$$\text{vol} = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\psi \, d\psi \, d\theta \, dr. = \frac{\pi}{6};$$