Lecture Notes

by Raphael Koch

23.11.2017

Def: Let $\Omega \subset \mathbb{R}^n$ be open. Ω is said to be (path) connected if for every pair of points $x, y \in \Omega$ there exists a C^1-path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$.

The following theorem gives a necessary condition for the existence of a potential function f for a vectorfield v (i.e. $v = \nabla f$).

Theorem (Satz 7.4.2 in Struwe): Let v be a continuous vectorfield on an open, connected set $\Omega \subset \mathbb{R}^n$. Then the following 3 statements are equivalent

1. v is the gradient of a potential function $f : \Omega \to \mathbb{R}$ (i.e. $v = \nabla f$).
2. The line integral of v is independent of the path. I.e. if $\gamma_1 : [a_1, b_1] \to \Omega$ and $\gamma_2 : [a_2, b_2] \to \Omega$ are two piecewise C^1-paths satisfying $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$, then
$$\int_{\gamma_1} v \; d\gamma_1 = \int_{\gamma_2} v \; d\gamma_2.$$
3. The line integral of v around any loop vanishes. I.e. if $\gamma : [a,b] \to \Omega$ is a piecewise C^1-path such that $\gamma(a) = \gamma(b)$ then
$$\int_{\gamma} v \; d\gamma.$$

Sketch of proof: It is easy to check that the second and third statements are equivalent. If $v = \nabla f$ and $\gamma : [a,b] \to \Omega$ then
$$\int_\gamma v \; d\gamma = \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t) \; dt = \int_a^b \frac{d}{dt} f(\gamma(t)) \; dt = f(\gamma(b)) - f(\gamma(a)).$$
In particular we see that the path integral only depends on the endpoints of the path. This proves that 1. implies 2. Assume condition 2. in the statement of the theorem. Then we need to show 1. This is done exactly as in the 1 dimensional case: Fix $p_0 \in \Omega$. Then we define
$$f(x) := \int_{\gamma_x} v \; d\gamma_x$$
where $\gamma_x : [0,1] \to \Omega$ is any C^1-path such that $\gamma_x(0) = p_0$ and $\gamma_x(1) = x$. By assumption f is well-defined. Now one needs to check that f is differentiable with $\nabla f = v$. For this we refer to Struwe's script. This finishes the sketch of the proof.

Def: A continuous vector field $v : \Omega \to \mathbb{R}$ is conservative if it has a potential (i.e. $\exists f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = v$)

Example: $v = \begin{bmatrix} y^2 \\ xz \\ 1 \end{bmatrix}$, $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^3$, $\gamma_1(t) = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$, $\gamma_2(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$, $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$.
$$\int_{\gamma_1} v d\gamma_1 = \frac{5}{3}, \int_{\gamma_2} v d\gamma_2 = \frac{23}{15}.$$

Indeed if $f : \mathbb{R}^3 \to \mathbb{R}$ would satisfy
$$\frac{\partial f}{\partial x} = -y^2, \; \frac{\partial f}{\partial y} = xz, \; \frac{\partial f}{\partial z} = 1 \Longrightarrow f(x,y,z) = -y^2 x + h(y,z)$$
$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial z} \Longrightarrow h(y,z) = z + g(y), \; f = -y^2 x + z + g(y)$$

Mads Bisgaard is entirely responsible for any typos or mistakes in these notes. Please write him at mads.bisgaard@math.ethz.ch if you spot a mistake.

but now we have

$$xz = \frac{\partial f}{\partial y} = -2yx + g'(y).$$

This cannot be solved for g, so v is not conservative.

Theorem: (Necessary condition for being conservative) Let $\Omega \subset \mathbb{R}^n$ be open and let $v : \Omega \to \mathbb{R}$ be a C^1 vectorfield ($v = (v_1, v_2, ..., v_n)$). If v is conservative then

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} \quad \forall\ 1 \leq i, j \leq n$$

Proof: If $v = \nabla f$ then f is in C^2, so $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Now use

$$v_i = \frac{\partial f}{\partial x_i} \implies \frac{\partial v_i}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Example: Let $\Omega := \mathbb{R}^2 \backslash \{(0,0)\}$ and consider the vectorfield $v : \Omega \to \mathbb{R}^2$,

$$v(x,y) = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix}$$

v is at least C^1. Moreover, it is an easy computation to check that

$$\frac{\partial v_1}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial v_2}{\partial x},$$

so v satisfies the condition in the above theorem. However, we saw in exercise 1a) on exercise sheet 8 that v is *not* conservative.

Exercise: $X = \{(x,y) \in \mathbb{R}^2 \,|\, x > 0\}$, $(x,y) = (r\cos\varphi, r\sin\varphi)$ with $r > 0$, $\varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})$ and $\tan\varphi = \frac{y}{x} \implies \varphi = \arctan(\frac{y}{x})$. Now compute that

$$\nabla\varphi = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix} = v$$

Hence, we conclude that v is conservative. But now note that v is exactly the same vectorfield as in the previous example where we concluded that v was not conservative! The crucial point here is that in the exercise v is defined on X and v is conservative on X, but it is not on Ω because the potential cannot be extended to all of Ω. The important lesson: **Being conservative is a property both of the vector-field and of the domain on which it is defined**.

Theorem: Let $\Omega \subset \mathbb{R}^n$ be open and convex. Let $v : \Omega \to \mathbb{R}^n$ be a C^1 vector field such that

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} \quad \forall 1 \leq i, j \leq n$$

Then v is conservative.

Note that in the Example above Ω is not convex. However, X is convex and we also checked in the exercise that v is conservative on X.

Integration in $\mathbb{R}^n$

Def: A (rectangular) box $Q \in \mathbb{R}^n$ is a set of type

$$Q = \prod_{i=1}^{n} I_i = \{(x_1, ..., x_n) \,|\, x_i \in I_i,\ \forall i \in \{1, ..., n\}\}$$

where $I_i = [a_i, b_i], i = 1, ..n.$ Hence $Q = [a_1, b_1] \times ... \times [a_n, b_n]$

$$Vol(Q) = \prod_{i=1}^{n} |b_i - a_i| = \mu(Q)$$

Def: A partition $P = \{Q_j\}_{j=1}^{l}$ of Q is a collection of boxes Q_j such that

1. $Q = \bigcup_{i=1}^{l} Q_i$

2. the interior of Q_i and Q_j do not intersect for $i \neq j$ ($Q_i \cap Q_j = \emptyset$)

We denote by $\mathcal{P}(Q)$ the set of all partitions of Q.

$$diam(Q_j) := \sup_{x,y \in Q_j} |x - y| \text{ (diameter of } Q_j\text{)}$$

Norm of partition P is $\delta_P := \max_{1 \leq j \leq l} diam(Q_j)$
If we have points $\xi_j \in Q_j$ then we have a Riemann sum for the partition P:

$$R_f(P) = \sum_{j=1}^{l} f(\xi_j) Vol(Q_j) = \sum_{j=1}^{l} f(\xi_j) \mu(Q_j).$$

Def: Let $f : Q \to \mathbb{R}$ be bounded. Let

$$U_f(P) = \sum_{j=1}^{l} \inf_{Q_j}(f) \mu(Q_j) \quad \text{(lower Riemann sum), "U" for "untere"}$$

$$O_f(P) = \sum_{j=1}^{l} \sup_{Q_j}(f) \mu(Q_j) \quad \text{(upper Riemann sum, "O" for "obere")}$$

Def: A refinement of a partition $P = \{Q_j\} \in \mathcal{P}(Q)$ is another partition $\widetilde{P} = \{\widetilde{Q}_k\}_{k=1}^{m} \in \mathcal{P}(Q)$ such that each $\widetilde{Q}_k$ is contained in some Q_j.

Lemma: If $\widetilde{P} \in \mathcal{P}(Q)$ is a refinement of $P \in \mathcal{P}(Q)$ then $U_f(\widetilde{P}) \geq U_f(P), O_f(\widetilde{P}) \leq O_f(P)$. If $P_1, P_2 \in \mathcal{P}(Q)$ then $U_f(P_1) \leq O_f(P_2)$.

Def: Let $f : Q \to \mathbb{R}$ be bounded. We define

$$\underline{I}(f) = \sup\{U_f(P) \,|\, P \in \mathcal{P}(Q)\} \quad \text{(lower Riemann integral)}$$
$$\overline{I}(f) = \inf\{O_f(P) \,|\, P \in \mathcal{P}(Q)\} \quad \text{(upper Riemann integral)}$$

We say that f is integrable if

$$\underline{I}(f) = \overline{I}(f).$$

If f is integrable then $I(f) := \underline{I}(f) = \overline{I}(f)$.

Lemma: $f : Q \to \mathbb{R}$ bounded is integrable if and only if $\forall \varepsilon > 0 \exists P \in \mathcal{P}(Q)$ such that $0 \leq O_f(P) - U_f(P) < \varepsilon$

Proof: Assume f is integrable, hence $\underline{I}(f) = \overline{I}(f)$. Let be $\varepsilon > 0$.
" $\Longrightarrow$ " We can choose $P', P'' \in \mathcal{P}(Q)$ such that $O \leq \underline{I}(f) - U_f(P') < \frac{\varepsilon}{2}$ and $0 < O_f(P'') - \overline{I}(f) < \frac{\varepsilon}{2}$. Let $P \in \mathcal{P}(Q)$ be a common refinement $O_f(P) - U_f(P) \leq O_f(P'') - U_f(P') < \varepsilon$.
" $\Longleftarrow$ " $U_f(P) \leq \underline{I}(f) \leq \overline{I}(f) \leq O_f(P) \quad \forall P \in \mathcal{P}(Q)$. In particular $0 < \overline{I}(f) - \underline{I}(f) \leq O_f(P) - U_f(P) \quad \forall P \in \mathcal{P}(Q)$. So, if for every $\epsilon > 0$ we can find a partition P such that $0 \leq O_f(P) - U_f(P) < \varepsilon$ is satisfied then $\underline{I}(f) = \overline{I}(f)$.

Given $P = \{Q_j\}_{j=1}^{l} \in \mathcal{P}(Q)$ we consider a step function

$$f(x) = \sum_{j=1}^{l} c_j \chi_{Q_j}(x).$$

where

$$\chi_{Q_j}(x) = \begin{cases} 1 & if\, x \in Q_j \\ 0 & if\, x \notin Q_j \end{cases}.$$

Then we have

$$\underline{I}(f) = \sum_{j=1}^{l} c_j Vol(Q_j) = \overline{I}(f) \Longrightarrow f \text{ is integrable}$$

Theorem: If $f : Q \to \mathbb{R}$ is continuous then f is integrable.

Proof: We use the "small span theorem". Let $\varepsilon > 0$. Since f is continuous the small span theorem gives the existence of a $P = \{Q_j\}_{j=1}^l \in \mathcal{P}(Q)$ such that

$$Span_f(Q_j) := sup_{Q_j}(f) - \inf_{Q_j}(f) < \frac{\varepsilon}{Vol(Q)} \quad \forall\, j \in \{1, \dots, l\}.$$

In particular

$$O_f(P) - U_f(P) = \sum_{j=1}^l \sup_{Q_j}(f)Vol(Q_j) - \sum_{j=1}^l \inf_{Q_j}(f)Vol(Q_j) = \sum_{j=1}^l Span_f(Q_j)Vol(Q_j) < \frac{\varepsilon}{Vol(Q)} \sum_{j=1}^l Vol(Q_j) = \varepsilon.$$

It now follows from the previous lemma that f is integrable. This finishes the proof.

More generally any $f : Q \to \mathbb{R}$ which is continuous except on a set of zero content is integrable. **Def:** Let X be a bounded subset of a plane, $X \subset Q$. The set X is said to have content zero if $\forall \varepsilon > 0$, $\exists Q_1, ..., Q_l$ boxes whose union contains X and whose sum of volumes doesn't exceed ε. I.e.

$$X \subset \bigcup_{j=1}^l Q_j, \quad \sum_{j=1}^l Vol(Q_j) \leq \varepsilon$$

Theorem: (small span theorem) If $f : Q \to \mathbb{R}$ is continuous then $\forall \varepsilon > 0\ \exists$ a partition $P = \{Q_j\}_{j=1}^l \in \mathcal{P}(Q)$ such that $Span_f(Q_j) < \varepsilon \forall j \in \{1, ..., l\}$, where $Span_f(Q) = \sup_Q(f) - \inf_Q(f)$

Theorem: Let $f, g : Q \subset \mathbb{R}^n \to \mathbb{R}$ be integrable and let $\alpha, \beta \in \mathbb{R}$. Then

1. $(\alpha f + \beta g) : Q \to \mathbb{R}$ is integrable we have linearity

$$\int_Q (\alpha f + \beta g) d\mu = \alpha \int_Q f d\mu + \beta \int_q g d\mu$$

2. if $f(x) \leq g(x) \forall\, x \in Q$ we have monotonity

$$\int_Q f d\mu \leq \int_Q g d\mu$$

3. if $f(x) \geq 0 \forall\, x \in Q$ we have positivity

$$\int_Q f d\mu \geq 0$$

4. (bounds)

$$\Big| \int_Q f d\mu \Big| \leq \int_Q |f| d\mu \leq \sup_Q |f| Vol(Q)$$

5. $P = \{Q_j\}_{j=1}^l \in \mathcal{P}(Q)$ such that $Q = \bigcup_{j=1}^l Q_j$ and $Q_j \cap Q_i = \emptyset \forall i \neq j$ then we have additivity

$$\int_Q f d\mu = \sum_{j=1}^l \int_{Q_j} f d\mu$$

Theorem: (Fubini) Let $Q = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^n$ and let $f : Q \to \mathbb{R}$ be continuous then

$$\begin{aligned} \int_Q f d\mu &= \int_{a_n}^{b_n} \Big(\int_{a_{n-1}}^{b_{n-1}} \Big(... \int_{a_1}^{b_1} f dx_1 ... \Big) dx_{n-1} \Big) dx_n \\ &= \int_{a_1}^{b_1} \Big(\int_{a_2}^{b_2} \Big(... \int_{a_n}^{b_n} f dx_n ... \Big) dx_2 \Big) dx_1. \end{aligned}$$

We can swap ranges: Example $g : Q^{(2)} \to \mathbb{R}$ with $Q^{(2)} = [a, b] \times [c, d]$. Then

$$\int_{Q^{(2)}} f d\mu = \int_c^d \int_a^b f dx dy = \int_a^b \int_c^d f dy dx$$

Exercise: Integral of $f(x, y) = 2x + 2yx = 2x(y + 1)$ on $Q = [0, 1] \times [-2, 2]$.

We see that the first interval is the domain of x and the second the domain of y. Hence we have

$$\int_Q f d\mu = \int_{-2}^{2}\int_0^1 f(x,y)dxdy = \int_{-2}^{2}\int_0^1 2x(y+1)dxdy = \int_{-2}^{2}\Big[x^2(y+1)\Big]_{x=0}^{1}dy = \int_{-2}^{2} y+1dy = \Big[\frac{1}{2}y^2+y\Big]_{y=-2}^{2} = 4$$

And similarly one can compute that

$$\int_0^1\int_{-2}^{2} f(x,y)dydx = 4.$$