28.9.17

Defn 1) $f : \mathbb{R}^n \to \mathbb{R}^m$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^m$

$$\lim_{x \to a} f(x) = b \iff \lim_{\|x-a\| \to 0} \|f(x) - b\| = 0$$

$\iff \forall \varepsilon > 0 \ \exists \delta > 0$ so that $\forall x \in \mathbb{R}^n$ with $\|x - a\| < \delta$ we have $\|f(x) - b\| < \varepsilon$.

2) f is continuous at $x = a \iff \lim_{x \to a} f(x) = f(a)$

Thm Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$, with $\lim_{x \to a} f(x) = b$, $\lim_{x \to a} g(x) = c$

Then
- $\lim f(x) + g(x) = b + c$
- $\lim \lambda f(x) = \lambda b$, $\forall \lambda \in \mathbb{R}$
- $\lim f(x) \cdot g(x) = b \cdot c$
- $\lim_{x \to a} \|f(x)\| = \|b\|$

Ex : • Linear Transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is cont. $\forall a \in \mathbb{R}^n$

• Polynomials $p: \mathbb{R}^n \to \mathbb{R}$ are cont $\forall a \in \mathbb{R}^n$

• Rational functions $f(x) = P(x)/Q(x)$ are cont $\forall a \in \mathbb{R}^n$, for which $Q(a) \neq 0$.

$P, Q: \mathbb{R}^n \to \mathbb{R}$ are polynomials.

Thm let f, g functions such that $f \circ g$ is defined at a. If g is cont at a and f is cont at $g(a)$ then $(f \circ g)$ is continuous at a.

Ex : 1) $f: \mathbb{R}^2 \to \mathbb{R}$

$$(x,y) \to \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist

2) $f: \mathbb{R}^2 \to \mathbb{R}$

$$(x,y) \to \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 . & (x,y) = (0,0) \end{cases}$$

$\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

- For studying limits of scalar fields the sandwich lemma can be useful

<u>Lemma</u> let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ where $f(x) < g(x) < h(x)$ $\forall x \in \mathbb{R}^n, a \in \mathbb{R}^n$

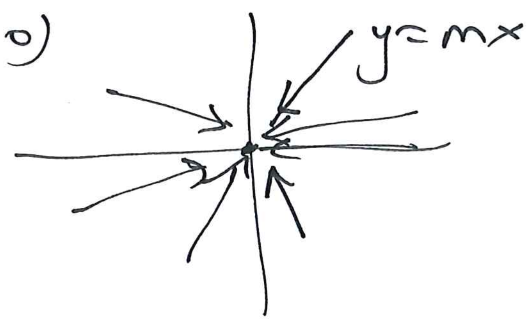
If $\lim\limits_{x \to a} f(x) = \lim h(x) = L$ then $\lim\limits_{x \to a} g(x) = L$.

- For studying limits in $\mathbb{R}^2$, it is also sometimes useful to use polar coordinates.

Clicker Question 1.

$(x,y) \neq (0,0)$

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^3+y^3} & (x,y) \neq (0,0) \\ 0 & \text{o.w.} \end{cases}$$



$$f(x,mx) = \frac{mx^3}{x^3+m^3x^3}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{m x^3}{x^3+m^3x^3} = \lim_{x\to 0} \frac{m}{1+m^3} = \frac{m}{1+m^3}$$

Since it depends on m, it ~~does~~ is not unique.

Hence the limit does not exist

$$\frac{m\cancel{x^3}}{\cancel{x^3}(m^3+1)} = \frac{m}{1+m^3}$$

Clicker Question 2

$\lim f(x,y) = L$ along every line $y = mx$

then $\lim f(x,y) = L$.

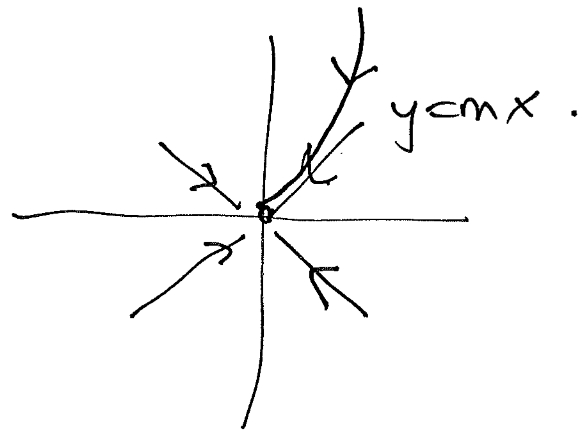
True

False

See the next clicker question

Clicker question 3.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & \text{o.w.} \end{cases}$$



$$\lim f(x,y) = 0 \quad ?$$

$$f(x, mx) = \frac{m x^3}{x^4 + m^2 x^2} = \frac{m x^3}{x^2 (x^2 + m^2)} = \frac{m x}{(x^2 + m^2)}$$

$$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{mx}{x^2 + m^2} = 0.$$

But along $y = \cancel{x}x^2$

$$\lim f(x, x^2) = \lim \frac{x^2 \cdot x^2}{x^4 + x^4} \; \cancel{\lim} \; \frac{x^4}{2x^4} = \frac{1}{2} \neq 0$$

- We have seen that in several variables, the existence of the limit along one direction is not enough for the existence of the limit.

- We'll see that existence of derivaties in a given direction does not guarantee the existence of "derivatives" or the differentiability of the function.

- What does the derivative in a direction mean?

§ Partial Derivatives and the differential:

We start with $f: \mathbb{R}^n \to \mathbb{R}$.

Recall: $f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$.

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

= instantaneous rate of change of f.

$$\frac{\Delta f}{\Delta x} = \text{average rate of change.}$$

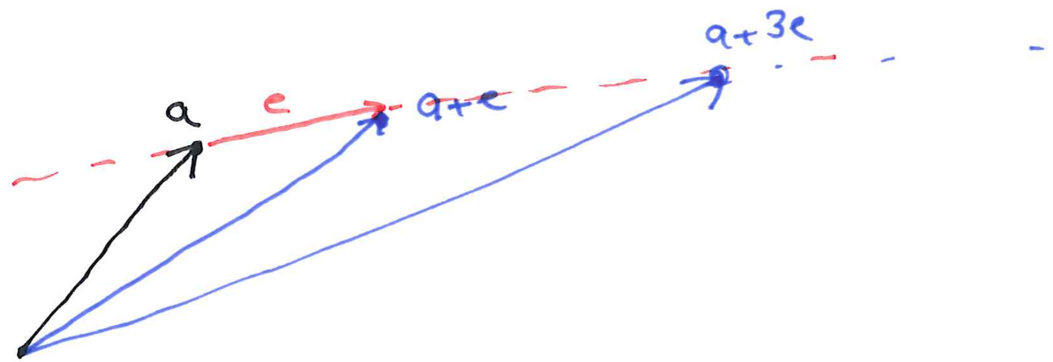
In general for $f = \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$

the manner in which f changes will depend on the direction in which we move away/to the point a.

We are interested in

$$\frac{f(\vec{a} + h\vec{e}) - f(\vec{a})}{h}.$$

as $h \to 0$.



__Defn__ (<u>Derivative of a scalar field with respect to a vector</u>)

Given $f : \Omega \to \mathbb{R}$ $\Omega \subseteq \mathbb{R}^n$, $\cancel{x}_0 \in \Omega$

Let $y \in \mathbb{R}^n$ an arbitrary fixed vector in $\mathbb{R}^n$.

The derivative of f at $\vec{x}_0$ with respect to y, denoted by $f'(\vec{x}_0;\vec{y})$ or $f'_y(\vec{x}_0)$, is defined as

$$f'(\vec{x}_0;\vec{y}) := \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{y}) - f(\vec{x}_0)}{h}$$

when the limit on the right exists.

$\boxed{\underline{\text{Example:}}}$ (Derivative of a linear transformation)

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a linear map.

(ie. $\forall \alpha, \beta \in \mathbb{R}$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, $f(\alpha\vec{x} + \beta\vec{y}) = \alpha f(\vec{x}) + \beta f(\vec{y})$.).

Take $y \in \mathbb{R}^n$ $\quad f'(x_0;y) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{y}) - f(x_0)}{h}$.

$$= \lim_{h \to 0} \frac{f(\vec{x}_0) + h f(\vec{y}) - f(\vec{x}_0)}{h} = \lim_{h \to 0} f(\vec{y}) = f(\vec{y}).$$

$\downarrow$
f linear.

For a linear function, the derivative wrt $\vec{y}$ at any point x_0 exists and is simply the value of f at $\vec{y}$.

<u>Remark</u> <u>Trivial case</u>. $f: \mathbb{R}^n \to \mathbb{R}$, if $\vec{y} = \vec{0}$

$$\frac{f(x_0 + hy) - f(x_0)}{h} = \frac{f(x_0) - f(x_0)}{h} = 0.$$

$$f'(x_0, \vec{0}) = 0.$$

We'll ~~so~~ restrict to vectors y of unit length, $\|y\|=1$. I'll write e instead of y to remind us that it is a vector of length 1.

In particular we are interested in the case

$$e = e_i = (0, \ldots, \underset{\uparrow \, i\text{-th.}}{1}, 0 \ldots 0) \in \mathbb{R}^n.$$

<u>Defn.</u> <u>Directional Derivative and partial derivatives</u>

Let $\vec{e} \in \mathbb{R}^n$ be a unit vector, $f : \mathbb{R}^n \to \mathbb{R}$, $\overline{x_0} \in \mathbb{R}^n$. The derivative $f'(x_0, \vec{e})$ is called the <u>directional derivative of f at x_0 in the direction e</u>. (Richtungsableitung von f im Punkt x_0 in Richtung $\vec{e}$).

When $e = e_i$ i-th coordinate vector $(0, 0, \ldots, 1, 0 \ldots)$ then the directional derivative $f(x_0, e_i)$ is called the <u>partial derivative</u> (wrt e_i), also denoted by $\frac{\partial f}{\partial x^i}(x_0)$, $D_i f(x_0)$, $f_i'(x_0)$

$$\frac{\partial f}{\partial x^i}(x_0) := \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{e}_i) - f(\vec{x}_0)}{h}$$

$x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$

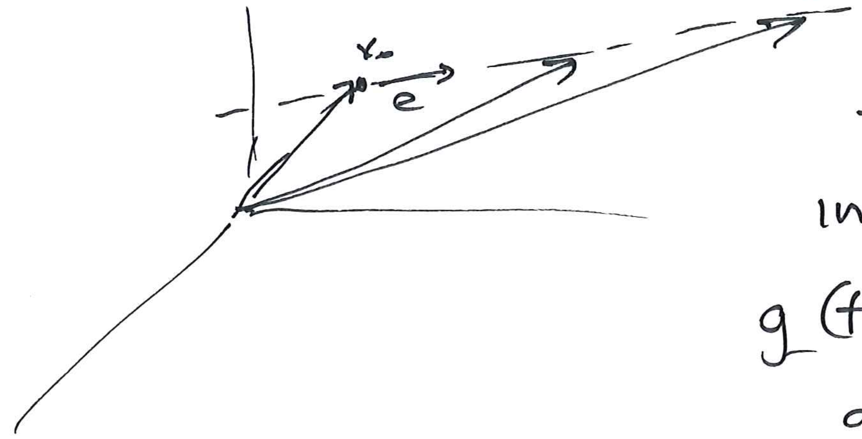
$x_0 + he_i = (x_0^1, x_0^2, \ldots, x_0^i + h, x_0^{i+1}, \ldots x_0^n)$

$$= \lim \frac{f(x_0^1, x_0^2, \ldots, x_0^i + h, x_0^{i+1}, \ldots x_0^n) - f(x_0^1, \ldots x_0^n)}{h}.$$

if the limit exists,

Remark. (1) The directional derivative of f at x_0 in the direction e, helps us understand how f behaves on the line passing through x_0 and $x_0 + e$

$\vec{x}_0 + \vec{e}t \qquad t \in \mathbb{R}$



To study this we can introduce a function

$$g(t) := f(\vec{x}_0 + \vec{e}t)$$

$$g : \mathbb{R} \to \mathbb{R}$$

Then $\quad g'(t) \text{ "=" } f'(x_0 + te, e)$ In particular

$$g'(0) = f'(x_0, e)$$

This is why in some books the directional derivative of f at $\bar{x_0}$ in the direction $\bar{e}$ is defined as.

$$d_e f(x_0) := \frac{d}{dt} f(x_0 + te)\Big|_{t=0.} = g'(0),$$

Why is it the case $\underline{g'(t) = f'(x_0+te, e)}$.

$$\lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\to 0.} \frac{f(\vec{x}_0 + t\vec{e} + h\vec{e}) - f(\vec{x}_0 + \vec{e}t).}{h.}$$

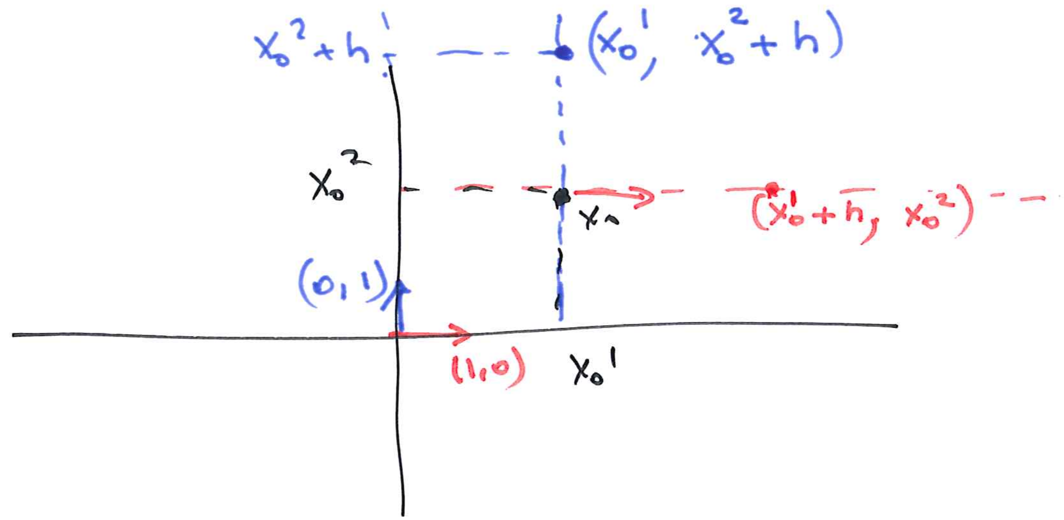
$$=: f'(x_0 + t\vec{e}, \vec{e}).$$

<u>Recall</u>. $f'(\vec{a}; \vec{e}) := \lim_{h\to 0} \frac{f(\vec{a} + h\vec{e}) - f(\vec{a})}{h.}$

$a := \vec{x}_0 + t\vec{e}$

2. In the case of partial derivatives

$$e = e_i = (0, \dots, 0, 1, 0, \dots, 0),$$



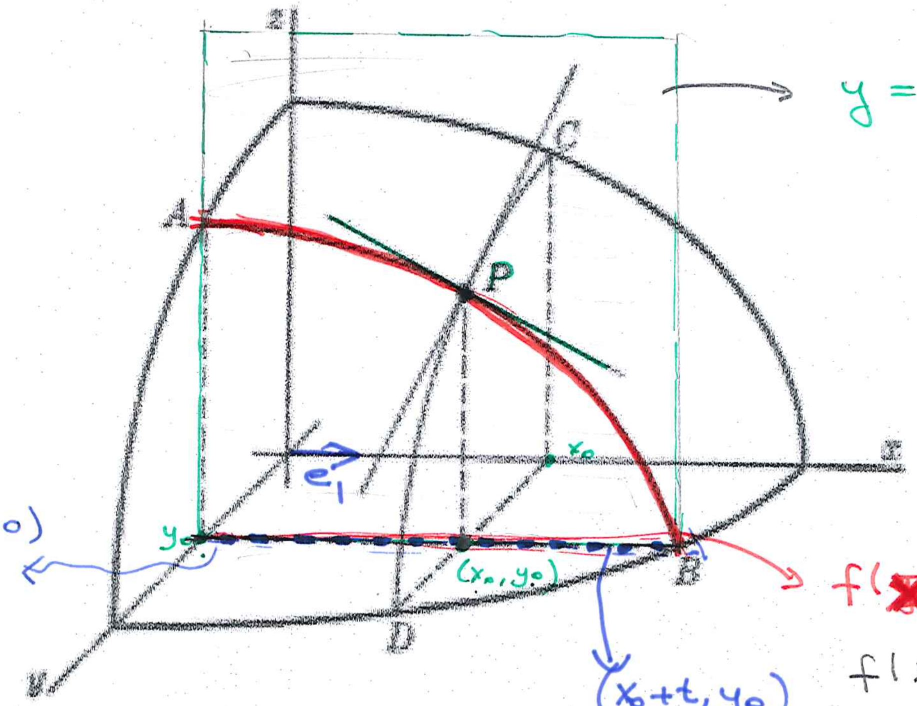
$$f : \mathbb{R}^2 \to \mathbb{R}.$$

$$\text{graf} = \{(x, y, z) \mid z = f(x,y)\}.$$

$z_0 = f(x_0, y_0)$

$P = (x_0, y_0, z_0)$

$y = y_0$ plane



$(x_0, y_0) + t(1,0)$ line

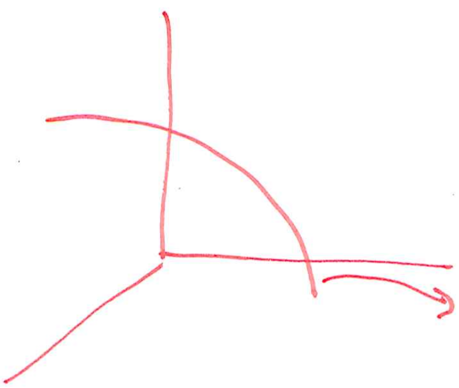
$f(\cancel{x}, y_0)$ curve

(x_0+t, y_0)

$f(x_0+t, y_0)$ curve.

Fig. 1

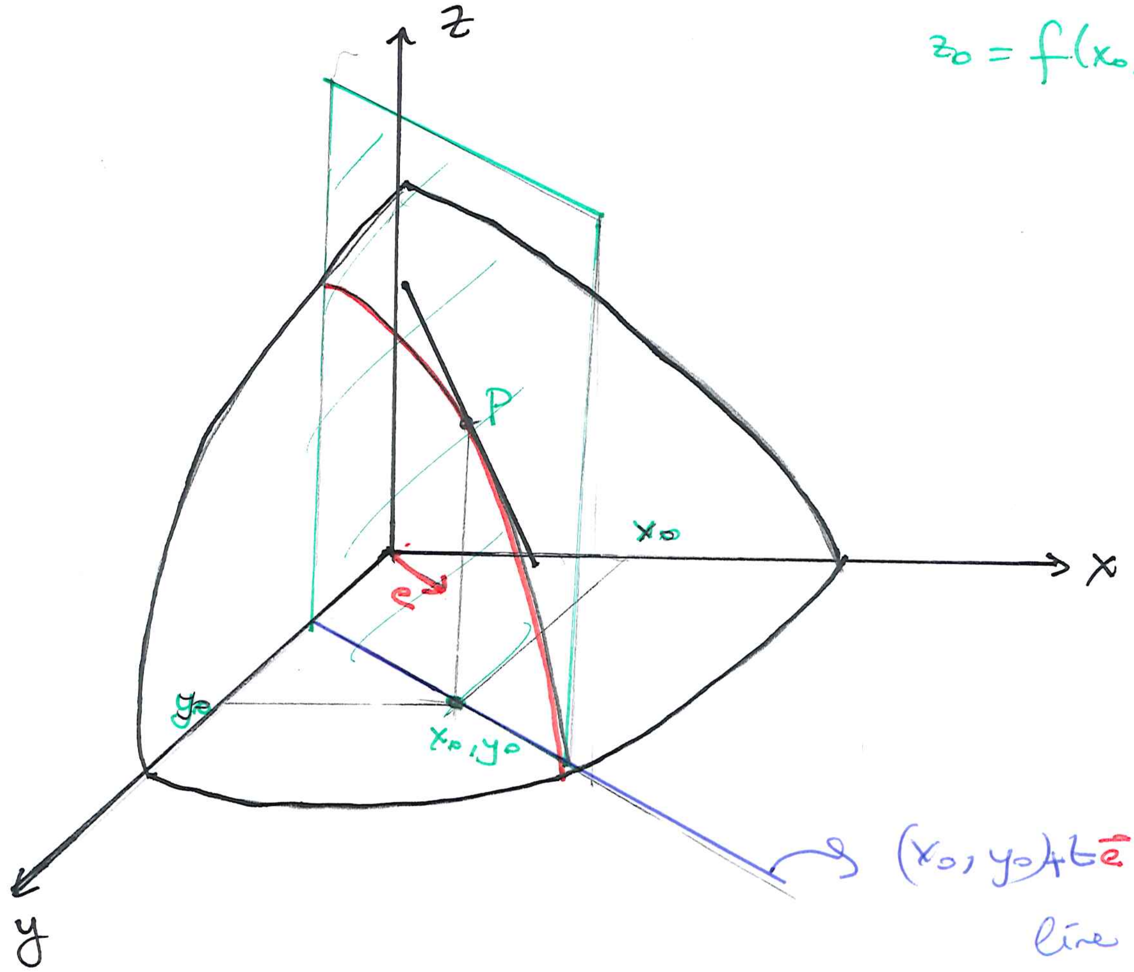
$$\text{slope} = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dt}\Big(f(x_0+t, y_0)\Big)\Big|_{t=0}.$$



$(x, y_0, f(x, y_0))$.

$P = (x_0, y_0, z_0)$

$z_0 = f(x_0, y_0)$



$$\text{slope} = f_e(x_0, y_0)$$
$$= d_e f(x_0, y_0)$$
$$= \frac{d}{dt} f((x_0, y_0) + t\vec{e})\Big|_{t=0}$$

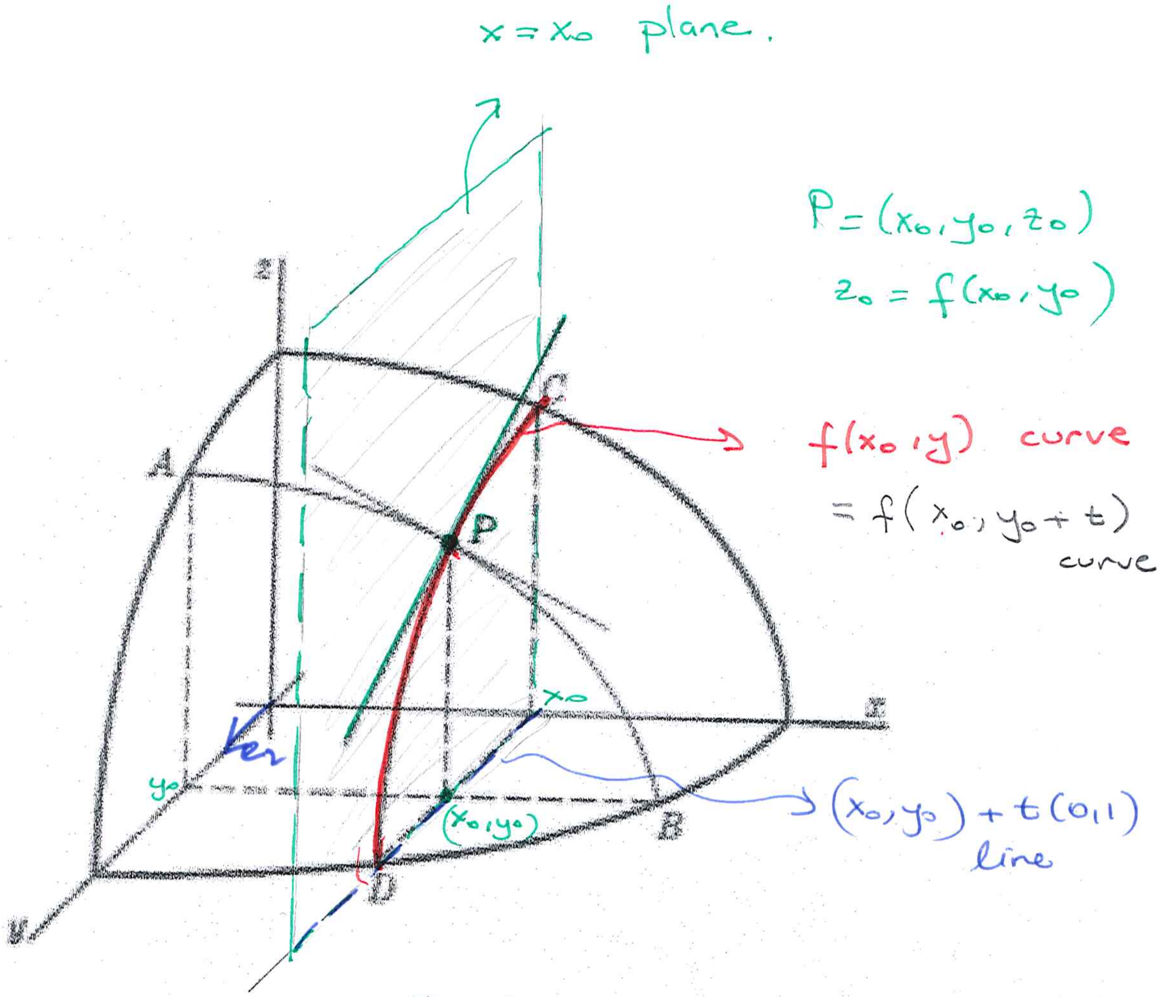


Fig. 1

$$\text{slope} = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{d}{dt}\left(f(x_0, y_0 + t)\right)\Big|_{t=0}.$$

Remark 3. To evaluate partial derivatives of a function say $f : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to f(x,y)$ one can for each fixed $y = y_0$ consider the function $f(x, y_0)$ as a function of one variable x, then differentiate with respect to x.

e.g. 1) $f : \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to (x^2 + yx)(\sin y)$

$$\frac{\partial f}{\partial x} = (\sin y)[2x + y]. \qquad \frac{\partial f}{\partial y} = x \sin y + (x^2 + yx)\cos y.$$

2) $f : \mathbb{R}^3 \to \mathbb{R}$.
$(x,y,z) \to e^{x^2 y} \cdot (x + y + z)$

$$\frac{\partial f}{\partial x} = \left(e^{x^2y} \cdot 2xy\right)(x+y+z) + e^{x^2y} \cdot 1.$$

$$\frac{\partial f}{\partial y} = \left(e^{x^2y} \cdot x^2\right)(x+y+z) + e^{x^2y} \cdot 1.$$

$$\frac{\partial f}{\partial z} = e^{x^2y} \cdot 1.$$

3) $f(x,y,z) = 3x^2yz + 2y^2z^3x + xy.$

$$\frac{\partial f}{\partial x} = 6xyz + 2y^2z^3 + y \qquad \frac{\partial f}{\partial z} = 3x^2y + 6y^2z^2x$$

$$\frac{\partial f}{\partial y} = 3x^2z + 4yz^3x + x$$