Thm Let v be a vector field, $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ continous on Ω. Ω open, connected sset of $\mathbb{R}^n$.

Then TFAE

1) v is the gradient of some potential function $f : \Omega \to \mathbb{R}$

ie $v = \nabla f$

2) The line integral of v is independent of the path in Ω.

i.e. if $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$, $\gamma_i : [a, b] \to \Omega$

Then $\displaystyle\int_{\gamma_1} v \cdot ds = \int_{\gamma_2} v \cdot ds$

3) $\displaystyle\oint_{\gamma} v \cdot ds = 0$ $\forall \gamma$, closed curve.

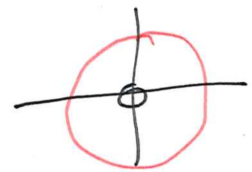
Defn. A vector field satisfying any of the above 3 equivalent properties is called conservative

Thm (Necessary condition for v to be conservative).

Let $v = (v_1, \dots v_n)$ be a continously diff. vector field on an open set $\Omega \subset \mathbb{R}^n$. If v is conservative then

$$\frac{\partial v_i}{\partial x^j} = \frac{\partial v_j}{\partial x^i} \qquad \forall\, i, j \quad , \quad 1 \le i, j \le n.$$

Rk Converse is not true!



eg. $\Omega = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$.

$v(x,y) = \left({-y}/{x^2+y^2} \, , \, {x}/{x^2+y^2} \right)$ satisfy $\dfrac{\partial v_1}{\partial y} = \dfrac{\partial v_2}{\partial x}$

but $\oint_\gamma v \, ds = 2\pi \neq 0$ if $\gamma = (\cos t, \sin t)$, $t \in [0, 2\pi]$.

__Rk__: But if $\Omega = \{(x,y) \mid x > 0\}$.

Then $\theta : \Omega \longrightarrow \mathbb{R}$ is a potential for v.
$(x,y) \longmapsto \arctan y/x$

ie $v = \nabla\theta$ on Ω.

__Hence__ Being conservative for a vector field v is a property of not only the vector field but also the region Ω, it is defined on!!

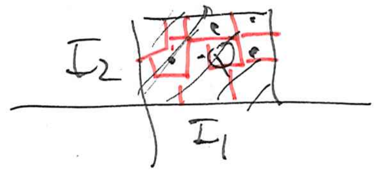
__Thm__ Let Ω be convex, and $v : \Omega \to \mathbb{R}^n$ a C^1 vector field.

Then v is conservative $\iff$ $\dfrac{\partial v_i}{\partial x^j} = \dfrac{\partial v_j}{\partial x^i} \quad \forall i,j$

Integration in $\mathbb{R}^n$.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Let $Q = \prod_{k=1}^{n} I_k = \{ x = (x^1, \dots, x^n) \mid x^k \in I_k, \ 1 \le k \le n \}$, $I_k = [a_k, b_k]$, intervals.



Vol $Q = \prod_{i=1}^{n} |b_i - a_i| = \mu(Q)$

A partition P of Q is a collection of subrectangular boxes $Q_1, \dots Q_k \in Q$ s.t

1) $Q = \bigcup_{j=1}^{k} Q_j$ and

2) $\mathrm{Int}(Q_i) \cap \mathrm{Int}(Q_j) = \emptyset$ for $i \neq j$.

Norm of P = mesh of P = $S_p := \mathrm{Max}(\mathrm{diam}\ Q_j)$. Let $f : \mathbb{R}^n \to \mathbb{R}$

For $\xi_j \in Q_j$, $P = \{Q_j\}_{j=1}^{k}$, $\boxed{R_f(P) := \sum_{j=1}^{k} f(\xi_j)\,\mathrm{vol}(Q_j)}$

is called the Riemann sum for the Partition P.

Lower Riemann sum: $U_f(P) := \sum_{j=1}^{k} \left(\operatorname{Inf}_{Q_j} f(x)\right) \mu(Q_j)$

Untere Summe

Upper Riemann sum $O_f(P) := \sum_{j=1}^{k} \left(\operatorname{Sup}_{Q_j} f(x)\right) \mu(Q_j)$

obere Summe

Let $f : Q \subset \mathbb{R}^n \to \mathbb{R}$ bdd.

Lower Riemann Integral $= \underline{I}(f) = \underline{\int_Q} f(x)\, d\mu := \operatorname{Sup}\{U_f(P) \mid P \in \mathcal{P}(Q)\}$.

Upper Riemann Integral $= \overline{I}(f) = \overline{\int_Q} f(x)\, d\mu := \operatorname{Inf}\{O_f(P) \mid P \in \mathcal{P}(Q)\}$.

f is called **integrable** if $\boxed{\overline{I}(f) = \underline{I}(f)}$.

<u>Lemma</u> f is integrable over $Q \iff \forall \varepsilon > 0 \ \exists P \in \mathcal{P}(Q)$ s.t.

$$O_f(P) - U_f(P) < \varepsilon.$$

<u>Defn</u> A step function on Q is a function f which is constant on each Q_j of some Partition P.

For a step function f, $\int_Q f =$ vol under the graph of f.

<u>Thm</u> Let f be defined and continuous on $Q = \prod_{i=1}^{n} I_i$

Then f is integrable on Q.

<u>Thm</u> Let f be bdd function on Q which is continuous except for finitely many points. Then f is integrable on Q

Thm $f, g : Q \subset \mathbb{R}^n \to \mathbb{R}$ R-integrable, $\alpha, \beta \in \mathbb{R}$. Then

1) $\alpha f + bg : Q \to \mathbb{R}$ is integrable and $\int_Q (\alpha f + \beta g)\, d\mu = \alpha \int_Q f + \beta \int_Q g$

2) if $f(x) \le g(x)$ $\forall x \in Q$ then

$$\int_Q f\, d\mu \le \int_Q g\, d\mu$$

3) if $f(x) \ge 0$ $\forall x \in Q$ then $\int_Q f\, d\mu \ge 0$.

4) $\left| \int_Q f\, d\mu \right| \le \int_Q |f|\, d\mu \le \left(\sup_Q |f|\right) \mathrm{vol}(Q)$

Thm (Fubini) $Q = [a,b] \times [c,d]$, f cont. on Q. Then

$$\int_Q f\, d\mu = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

Ex.: $f = e^x \sin y$ $\qquad Q = [0,1] \times [0, 2\pi]$.

$$\int_Q f \, d\mu = \iint_Q f \, dm = \int_0^1 \int_0^{2\pi} e^x \sin y \, dy \, dx$$

$$= \int_0^{2\pi} \int_0^1 e^x \sin y \, dx \, dy.$$

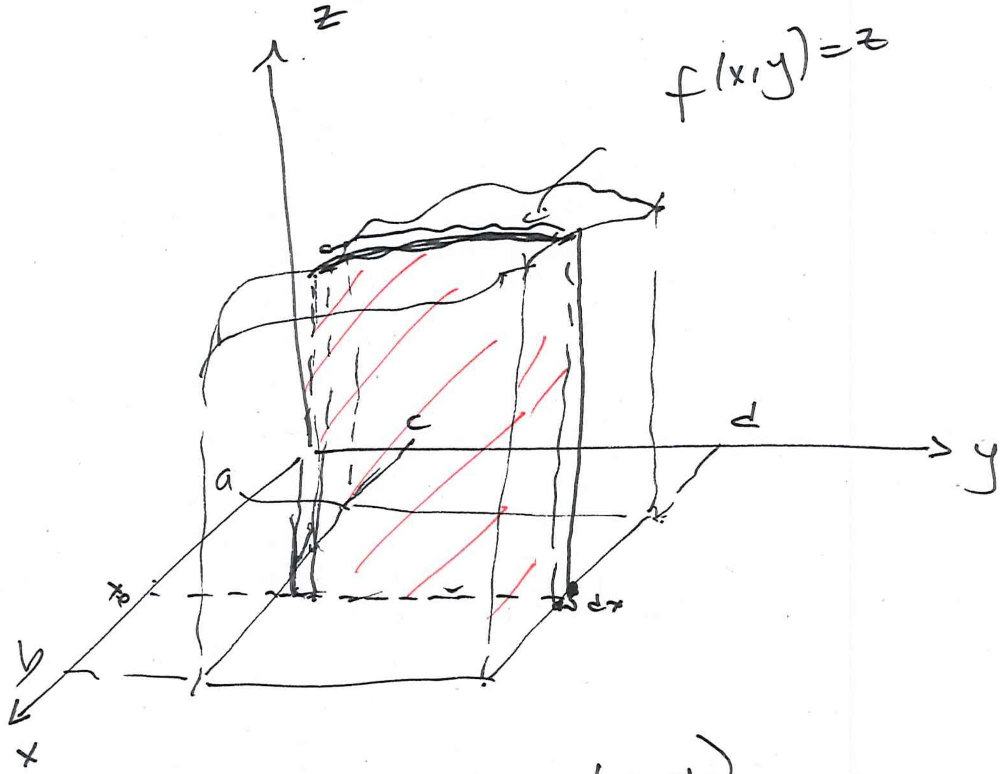
$$\int_0^1 \left(\int_0^{2\pi} e^x \sin y \, dy \right) dx = \int_0^1 e^x \Bigg(\underbrace{\int_0^{2\pi} \sin y \, dy}_{-\cos y \big|_0^{2\pi}} \Bigg) dx = \int_0^1 e^x \cdot 0 \, dx$$

$$= 0.$$

Geometric Interpretation.

$D = [a,b] \times [c,d]$.

$f : \mathbb{R}^2 \to \mathbb{R}$.



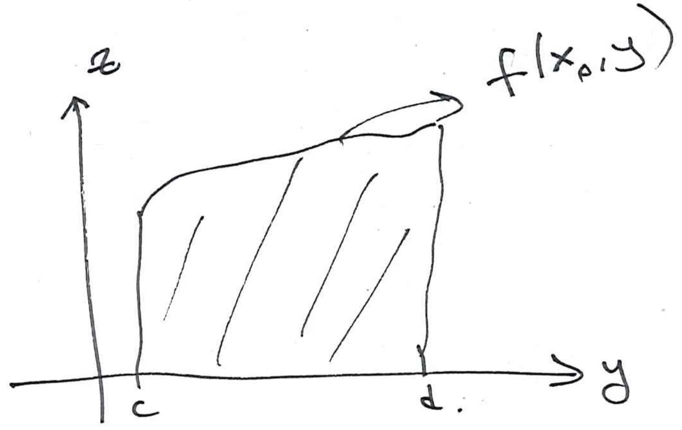
cross section has an Area

$$= \int_c^d f(x_0,y)\,dy = A(x_0)$$

vol of each tiny 'slice' is $A(x_0)\,dx$

volume of the small slice

$$\left(\int_c^d f(x_0,y)\,dy\right) dx$$



Sum these slices together to find the total volume under the graph of f

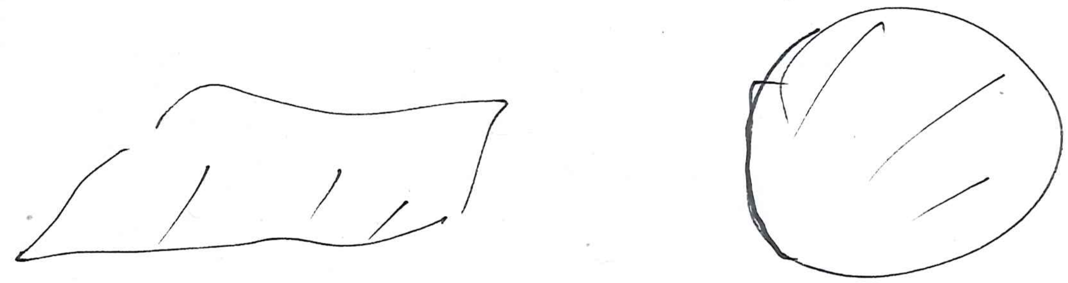
$V =$ sum of volumes of chips.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

We could also use cross sections parallel to xz plane instead and add those together.

$$V = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Until now we only looked at integrals of f over rectangles. But typically the integration region is not a rectangle.

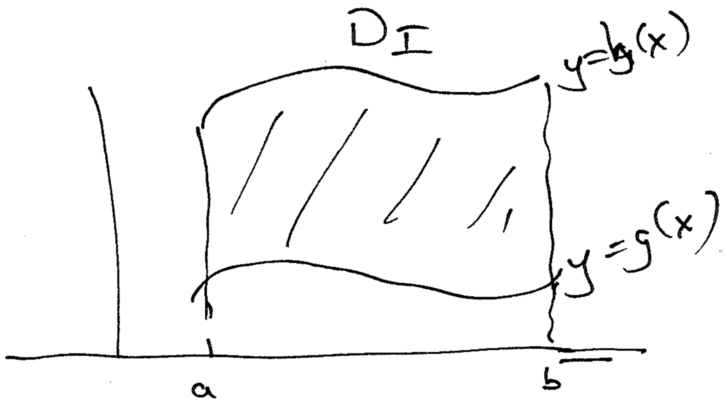


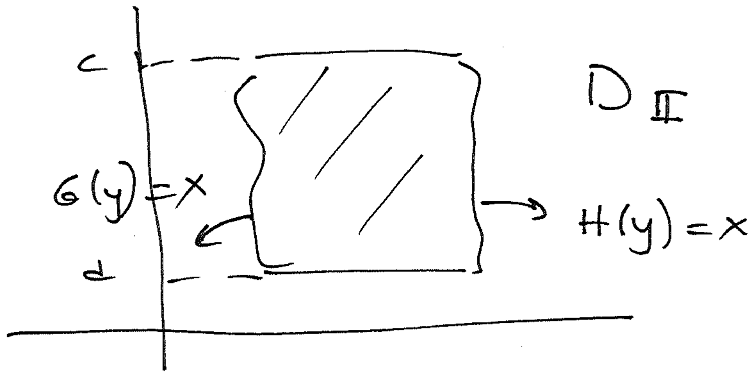
Double integrals over more general regions

Defn A subset $D \subset \mathbb{R}^2$ is called a normal region with respect to the x-axis (resp. y-axis) if there are 2 continuous functions g, h (resp G, H) such that

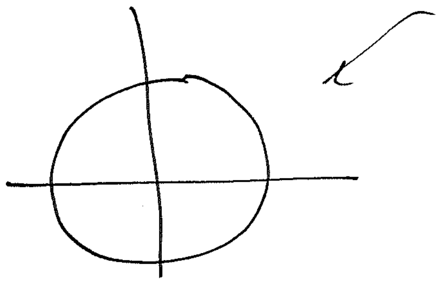
$$D = D_I = \{(x,y) \mid a \le x \le b \text{ and } g(x) \le y \le h(x)\}.$$

$$\left(\text{resp. } D = D_{II} = \{(x,y) \mid c \le y \le d \text{ and } G(y) \le x \le H(y)\}\right)$$

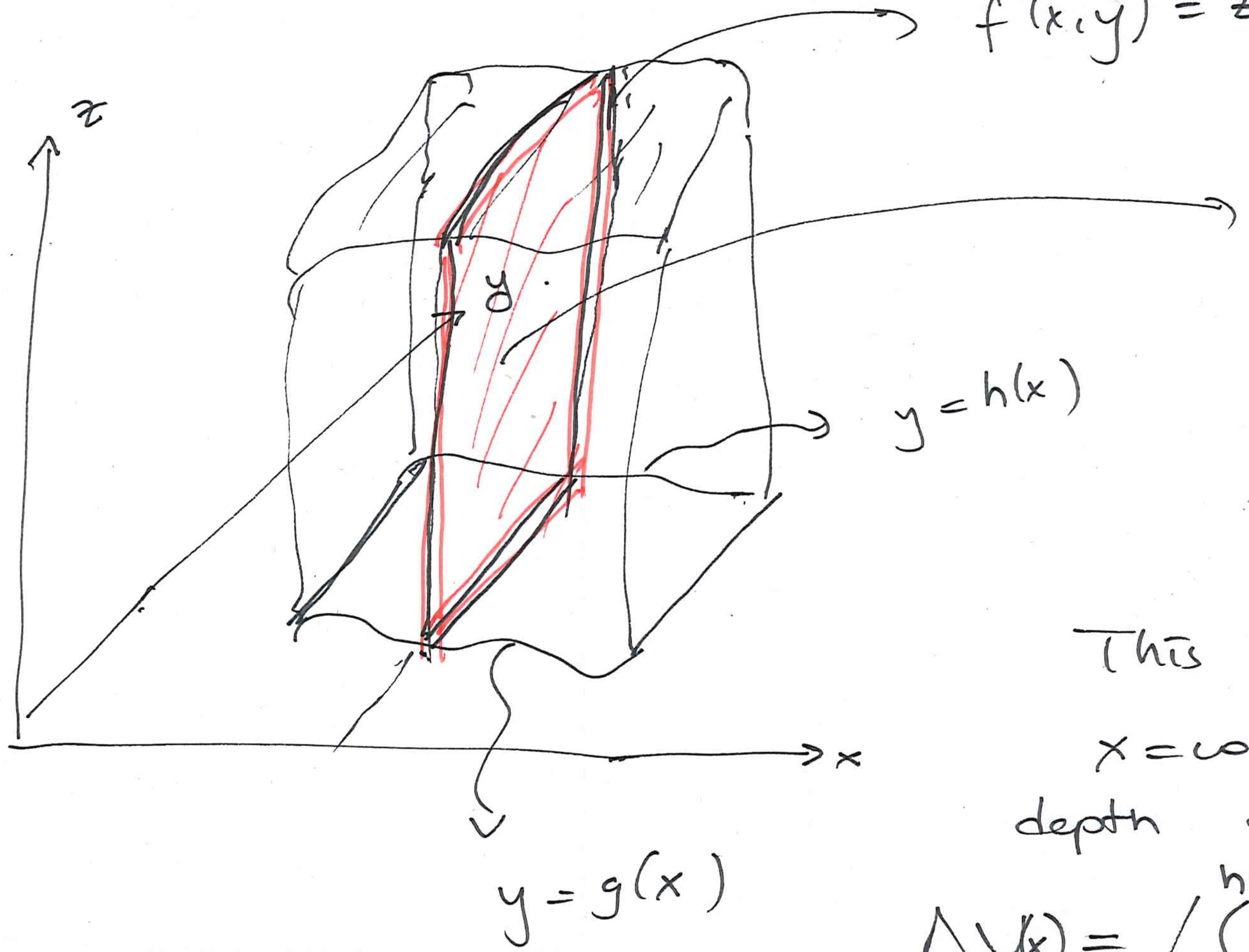


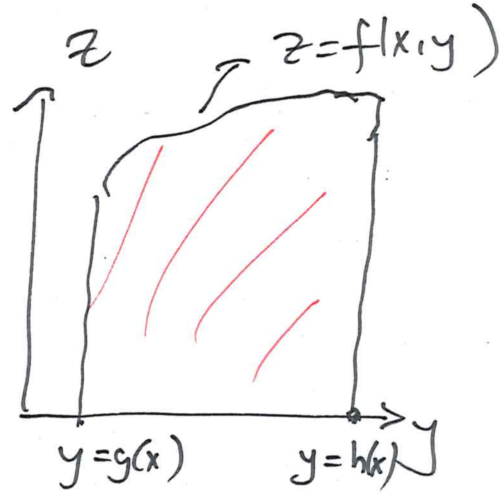


D is called of type III if it is simult. of type I and II.



~~As~~ type III.





This cross section for $x = \text{constant}$ with small depth dx has volume

$$\Delta V(x) = \left(\int_{g(x)}^{h(x)} f(x,y)\, dy \right) dx$$

to get the volume we sum $\Delta V(x)\, dx$ over $x \in [a,b]$.

$$V = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\, dy \right) dx .$$

Thm (Fubini) let f be continuous on a sset D of $\mathbb{R}^2$

(a) Suppose D is a normal region with respect to x.

$$D = \{ (x,y) \in \mathbb{R}^2 \mid a \le x \le b , \ g(x) \le y \le h(x) \}$$

for continuous g, h. Then f is integrable on D

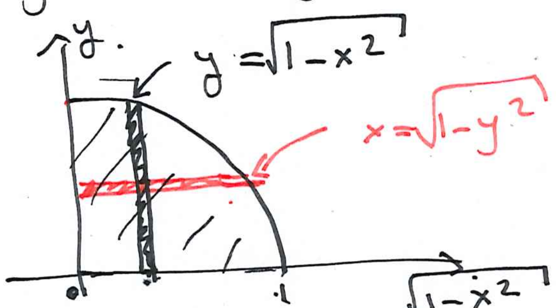
and
$$\int_D f \, d\mu = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y) \, dy \right) dx .$$

(b) Suppose D is a normal region w.r.t. y

$$D = \{ (x,y) \mid c \le y \le d \quad G(y) \le x \le H(y) \}.$$

then f is integrable on D and
$$\int_D f \, d\mu = \int_c^d \left(\int_{G(y)}^{H(y)} f(x,y) \, dx \right) d\cancel{x}y$$

Ex. Let $f(x,y) = x-y$, D be the region given by the quarter of a unit disc



$$\int_D f\,d\mu = ?$$

$$\int_0^1 \left(\int_0^{\sqrt{1-x^2}} (x-y)\,dy \right) dx = \int_0^1 \left(xy - \frac{y^2}{2} \Big|_{y=0}^{y=\sqrt{1-x^2}} \right) dx$$

$$= \underbrace{\int_0^1 x\sqrt{1-x^2}\,dx}_{} - \underbrace{\int_0^1 \frac{(1-x^2)}{2}}_{} \, dx = 0.$$

$u = 1-x^2$
$du = -2x\,\cancel{dx}$..

$$= -\int_1^0 \frac{u^{1/2}}{2} \qquad \underbrace{\frac{1}{2}\left(x - \frac{x^3}{3}\right)\Big|_0^1}_{\frac{1}{3}}.$$

$$= \frac{1}{2}\int_0^1 u^{1/2}\,du.$$

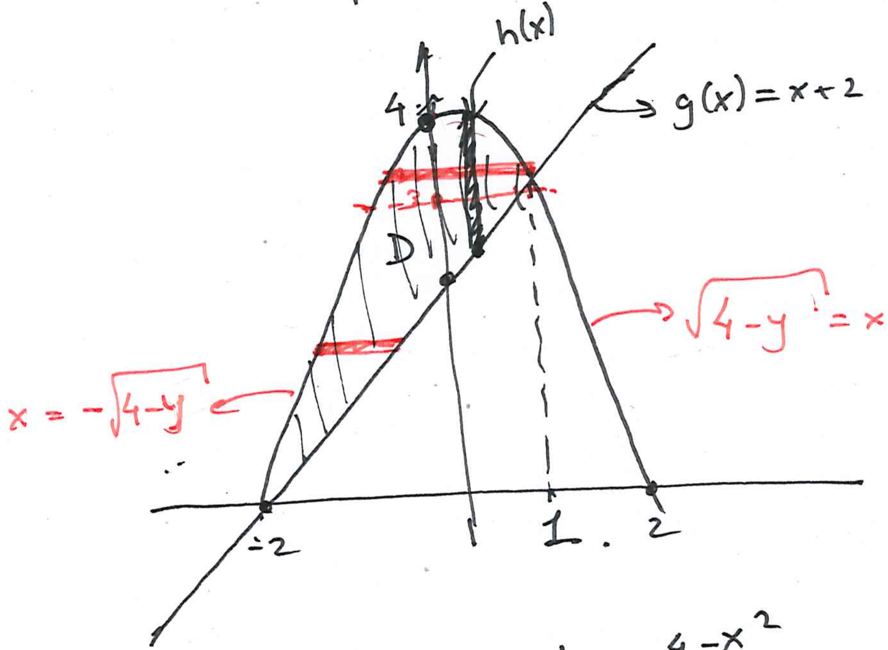
$$= \frac{1}{2}\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{1}{3}.$$

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x-y)\,dx \; dy = 0.$$

(2) let $f(x,y) = x$, D be the region bounded by the line $g(x) = x+2$ and the parabola $h(x) = 4-x^2$. $\int_D f \, d\mu = ?$

$= (2-x)(2+x)$

$y = 4-x^2$
$x^2 = 4-y$ $\Rightarrow x = \mp\sqrt{4-y}$.



We first find the intersection points.

$4-x^2 = x+2$.

$x^2+x-2=0$

$(x+2)(x-1)=0 \Rightarrow x=-2$
$= 1$

$$\int_D f \, d\mu = \int_{-2}^{1} \left(\int_{x+2}^{4-x^2} x \, dy \right) dx = \int_{-2}^{1} \left(xy \Big|_{y=x+2}^{y=4-x^2} \right) dx$$

$$= \int_{-2}^{1} x(4-x^2) - x(x+2) \, dx = \ldots$$

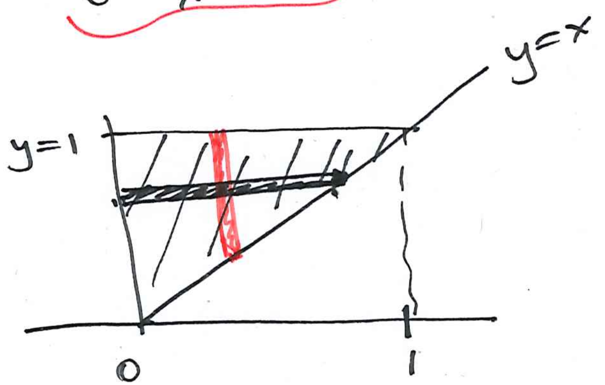
If we want to integrate in x first, ie.
if we want to divide the region D into sum of horizontal rectangles parallel to the x-axis, then we need to divide the region D into 2 pieces.

$$\int_D f\,d\mu = \int_0^3 \int_{-\sqrt{4-y}}^{y-2} x \,dx\,dy + \int_3^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} x \,dx\,dy$$

$$= \ldots$$

3) $\displaystyle\int_0^1 \int_x^1 e^{y^2}\,dy\,dx = ?$

$\int e^{y^2} dy \leftarrow$ e^{y^2} does not have an explicit primitive



$$\int_0^1 \int_{x=0}^{x=y} e^{y^2}\,dx\,dy = \int_0^1 \left(\int_0^y e^{y^2}\,dx \right) dy.$$

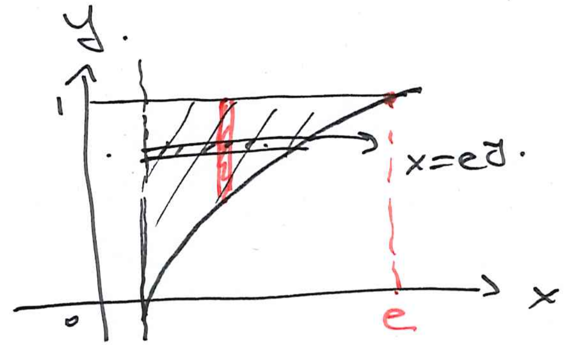
$$= \int_0^1 \left(e^{y^2} x \Big|_{x=0}^{x=y} \right) dy = \int_0^1 y e^{y^2}\,dy.$$

$$= \frac{e^{y^2}}{2}\Big|_0^1 = \frac{1}{2}(e-1).$$

Rk: Sometimes one <u>has</u> to change the order of integration to be able to calculate the integral.

Ex Change the order of integration in

$$\int_0^1 \int_{1=x}^{e^y=x} f \, dx \, dy . = \int_1^e \int_{?\, y=\ln x}^{y=1 .} f \, dy \, dx$$



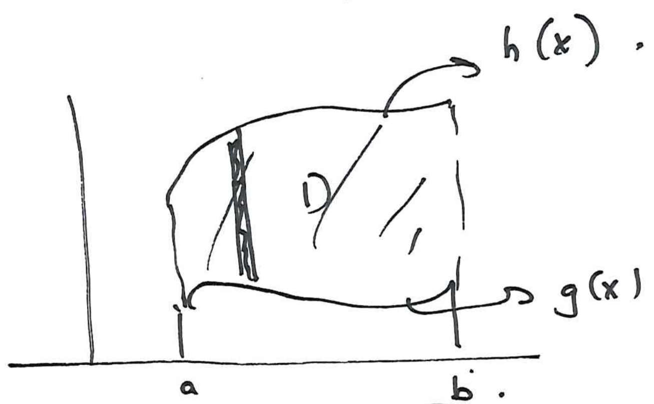
$x = e^y$

$\ln x = y$

Applications of double integrals.

1) Let D be a normal region w.r.t x_axis.

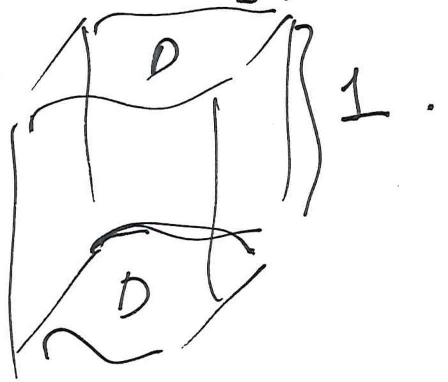
Then $\int_D 1 \, d\mu$ gives the area of D,



$$\int_D 1 \, d\mu = \int_a^b \int_{g(x)}^{h(x)} dy \, dx.$$

$$\int_a^b \big(h(x) - g(x) \big) \, dx$$

$$= \text{Area of } D.$$



(2). Let D be a thin plate in $\mathbb{R}^2$ with matter distributed with density $f(x,y)$ (mass / unit area).

The mass of D is given by

$$m(D) = \iint_D f(x,y)\, d\mu .$$

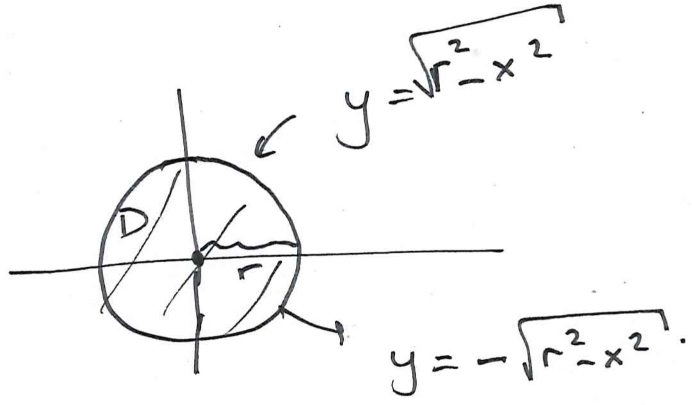
The average density is $\dfrac{m(D)}{\text{area}(D)} = \dfrac{\iint_D f(x,y)\, dx\, dy}{\iint_D 1\, dx\, dy}$.

Center of mass of D is given by $(\bar{x}, \bar{y})$

where $\bar{x} = \dfrac{1}{m(D)} \iint_D x f(x,y)\, d\mu$, $\bar{y} = \dfrac{1}{m(D)} \iint_D y f(x,y)\, d\mu$.

When the density is constant $=1$, the center of mass is called the centroid of D.

Ex. Centroid of a circular region of radius r is at the center. Because:



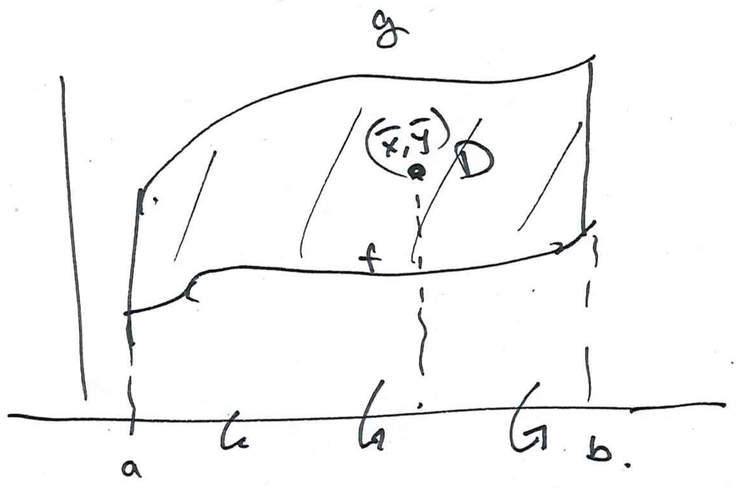
$$\bar{x}(D) = \frac{1}{A(D)} \iint_D x \, dx\, dy.$$

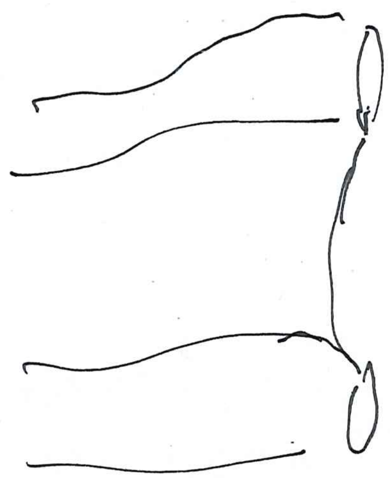
$$\bar{y}(D) \quad \iint_D y \, dx\, dy.$$

$$\bar{x}(D) = \frac{1}{A(D)} \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} x \, dy \, dx = \cdots = 0.$$

$$\bar{y}(D) = \quad - \quad — \quad = 0.$$

Theorem of Pappus. Let D be the region lying between the graphs of 2 continuous functions f and g over an interval $[a, b]$ where $0 \le f \le g$.





Let S be the solid of revolution generated by rotating D about the x-axis

Let V be the volume of this solid and $(\bar{x}, \bar{y})$ the centroid of D. Then

$$\boxed{V(S) = 2\pi \bar{y} A(D)}.$$

I.e. Pappus' theorem relates the volume of solid of revolution to the centroid of the region.