Aufgabe I.

(a) Mit $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\sinh(t)$ gilt

$$\int \frac{dx}{(2x+1)\sqrt{x^2+x+1}} = \int \frac{dt}{\sqrt{3}\sinh(t)} \underset{u=\cosh(t)}{=} \frac{1}{\sqrt{3}}\int \frac{du}{u^2-1} = \frac{1}{2\sqrt{3}}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$$
$$= \frac{1}{2\sqrt{3}} \log\left|\frac{u-1}{u+1}\right| + C$$
$$= \frac{1}{2\sqrt{3}} \log\left|\frac{2\sqrt{x^2+x+1}-\sqrt{3}}{2\sqrt{x^2+x+1}+\sqrt{3}}\right| + C.$$

(b) Es gilt

$$\int \frac{\cosh(x)}{\cosh(x)+\sinh(x)} dx \underset{t=\tanh(x)}{=} \int \frac{du}{(1-u)(1+u)^2} = \frac{1}{4}\log\left|\frac{\tanh(x)+1}{\tanh(x)-1}\right| - \frac{1}{2}\frac{1}{1+\tanh(x)} + C.$$

(c) Es gilt

$$\int \frac{dx}{\cosh^3(x)} = \int \frac{\cosh(x)dx}{(1+\sinh^2(x))^2} \underset{t=\sinh(x)}{=} \int \frac{dt}{(1+t^2)^2} = \frac{1}{2}\arctan\sinh(x) + \frac{1}{2}\frac{\sinh(x)}{\cosh^2(x)} + C.$$

Aufgabe II.

(a) Es gilt

$$\sum_{k=1}^{n} \frac{n^2}{(n+2k)^3} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{(1+\frac{2k}{n})^3} \underset{n\to\infty}{\longrightarrow} \int_0^1 \frac{dx}{(1+2x)^3} = \left[-\frac{1}{4(1+2t)^2}\right]_0^1 = \frac{2}{9}.$$

(b) Wenn $a > 1$ gilt

$$\sum_{k=1}^{\infty} \frac{1}{k^a} < \infty$$

also

$$\sum_{k=n+1}^{2n} \frac{1}{k^a} \leq \sum_{k=n+1}^{\infty} \frac{1}{k^a} \underset{n\to\infty}{\longrightarrow} 0.$$

Foglich gilt

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \underset{n\to\infty}{\longrightarrow} \int_0^1 \frac{dt}{1+t} = \log(2).$$

(c) Für alle $x \geq 0$ gilt

$$x - \frac{x^3}{6} \leq \sin(x) \leq x$$

also

$$\left|\sum_{k=n+1}^{2n} \sin\left(\frac{1}{k}\right) - \sum_{k=n+1}^{2n} \frac{1}{k}\right| \leq \frac{1}{6}\sum_{k=n+1}^{2n} \frac{1}{k^3} \underset{n\to\infty}{\longrightarrow} 0$$

und

$$\lim_{n\to\infty} \sum_{k=n+1}^{2n} \sin\left(\frac{1}{k}\right) = \log(2).$$

III.

(a) Es gilt

$$I_n - I_{n-1} = -\int_0^\infty \frac{x^2 dx}{(1+x^2)^{n+1}} = -\left[-\frac{x}{2n(1+x^2)^n}\right] - \frac{1}{2n}\int_0^\infty \frac{dx}{(1+x^2)^n} = -\frac{1}{2n} I_{n-1}$$

also

$$I_n = \frac{2n-1}{2n} I_{n-1} = \frac{(2n)!}{2^{2n}(n!)^2} I_0 = \frac{(2n)!}{2^{2n+1}(n!)^2}\pi.$$

(b) Wir haben nach $t \sim 0$

$$\frac{e^{-at} - e^{-bt}}{t} = \frac{(1 - at + O(t^2)) - (1 - bt + O(t^2))}{t} = b - a + O(t)$$

also

$$\int_0^1 \left|\frac{e^{-at} - e^{-bt}}{t}\right| dt = \int_0^1 \frac{e^{-at} - e^{-bt}}{t} < \infty \tag{1}$$

und

$$\int_1^\infty \left|\frac{e^{-at} - e^{-bt}}{t}\right| \le \int_1^\infty e^{-at} dt = \frac{1}{ae} < \infty \tag{2}$$

Wegen (1) und (2) existiert die Integral. Da

$$\lim_{t \to 0} t \log(t) = 0 \quad \text{und} \quad \lim_{t \to \infty} \log(t) e^{-t} = 0$$

gilt folglich

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \int_0^\infty \log(t)\left(ae^{-at} - be^{-bt}\right) dt. \tag{3}$$

Dann gilt für alle $a > 0$

$$\int_0^\infty |\log(t) e^{-at}| dt < \infty$$

und für alle $b > 0$

$$\begin{aligned}\int_0^\infty \log(t) e^{-bt} dt \underset{bt=au}{=} \frac{a}{b}\int_0^\infty \log\left(\frac{a}{b}u\right) e^{-au} du &= \frac{a}{b}\log\left(\frac{a}{b}\right)\int_0^\infty e^{-au} du + \frac{a}{b}\int_0^\infty \log(u) e^{-au} du \\ &= -\frac{1}{b}\log\left(\frac{b}{a}\right) + \frac{a}{b}\int_0^\infty \log(u) e^{-au} du. \end{aligned} \tag{4}$$

Damit folgt wegen (3) und (4)

$$\begin{aligned}\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt &= \int_0^\infty \log(t)\left(ae^{-at} - be^{-bt}\right) dt \\ &= \int_0^\infty a \log(t) e^{-at} dt - b\left(-\frac{1}{b}\log\left(\frac{b}{a}\right) + \frac{a}{b}\int_0^\infty \log(u) e^{-au} du\right) \\ &= \log\left(\frac{b}{a}\right).\end{aligned}$$

Variante 1. Sei

$$f(b) = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt.$$

Denn ist f differenzierbar über $(0,\infty)$ und gilt für alle $b>0$

$$f'(b) = \int_0^\infty e^{-bt}dt = \frac{1}{b}$$

Da $f(a)=0$ gilt folglich für alle $b>0$

$$f(b) = \int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt = \log\left(\frac{b}{a}\right).$$

Variante 2. Es gilt wegen Fubini Satz und $e^{-tu}>0$ für alle $t,u\geq 0$

$$\int_a^b \frac{e^{-at}-e^{-bt}}{t}dt = \int_0^\infty \left(\int_a^b e^{-tu}du\right)dt = \int_a^b \frac{du}{u} = \log\left(\frac{b}{a}\right).$$

(c) Es gilt

$$\int_0^2 \frac{dx}{\sqrt{x+1}(x+4)} \underset{y=\sqrt{1+x}}{=} \int_1^{\sqrt{3}} \frac{1}{y(y^2+3)}2ydy = \int_1^{\sqrt{3}} \frac{2dy}{y^2+3} = \left[\frac{2}{\sqrt{3}}\arctan\left(\frac{y}{\sqrt{3}}\right)\right]_1^{\sqrt{3}}$$
$$= \frac{2}{\sqrt{3}}\left(\arctan(1) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}.$$

wegen die Identität

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \text{ für alle } x>0$$

und

$$\arctan(\sqrt{3}) = \frac{\pi}{3}.$$

(d) Es gilt

$$\frac{d}{dt}\left(\arcsin\left(\frac{2t}{1+t^2}\right)\right) = \left(\frac{2}{1+t^2} - \frac{4t^2}{(1+t^2)^2}\right)\frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}} = \frac{2}{1+t^2}\frac{1-t^2}{|1-t^2|} \quad (5)$$

und

$$\int_0^1 \arcsin\left(\frac{2t}{1+t^2}\right)dt = \left[t\arcsin\left(\frac{2t}{1+t^2}\right)\right]_0^1 - \int_0^1 \frac{2t}{1+t^2}dt = \frac{\pi}{2} - \log(2).$$

Dann gilt

$$\begin{aligned}\int_1^{\sqrt{3}} \arcsin\left(\frac{2t}{1+t^2}\right)dt &= \left[t\arcsin\left(\frac{2t}{1+t^2}\right)\right] + \int_1^{\sqrt{3}}\frac{2t}{1+t^2}dt \\ &= \sqrt{3}\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(1) + \log(2) \\ &= \frac{\pi}{\sqrt{3}} - \frac{\pi}{2} + \log(2). \quad (6)\end{aligned}$$

Folglich gilt wegen (5) und (6)

$$\int_0^{\sqrt{3}} \arcsin\left(\frac{2t}{1+t^2}\right)dt = \frac{\pi}{\sqrt{3}}.$$