Analysis III for D-BAUG, Fall 2017 — Lecture 12

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1 The dynamic beam equation

Recall the static beam equation for the deflection curve $y(x)$:

$$EIy''''(x) + ky(x) = f(x),$$

where E is a material constant, I is the moment of inertia of a cross-section of the beam, k is the modulus of the elastic foundation, and $f(x)$ is an external load. Let's make a trivial but informative re-arrangement of this equation:

$$0 = -EIy''''(x) - ky(x) + f(x),$$

which we interpret as saying that the net force acting on the beam is zero. This means that the beam is in equilibrium. In the dynamic case this is no longer true. The net force is not zero, which leads to acceleration and a time-dependent deflection curve $y = y(x,t)$. It satisfies the PDE

$$y_{tt}(x,t) = -\frac{EI}{\rho} y_{xxxx}(x,t) - \frac{k}{\rho} y(x,t) + \frac{f(x)}{\rho},$$

where ρ is the mass per unit length. As in the static case, the boundary conditions depend on how the beam is attached. For example, if the beam is simply supported at both ends, the boundary conditions are

$$y(0,t) = y_{xx}(0,t) = y(L,t) = y_{xx}(L,t) = 0, \qquad t \geq 0.$$

Additionally, as usual for PDE, we must specify initial conditions $y(x,0)$ and $y_t(x,0)$ corresponding to the initial shape and initial velocity of the beam. In this lecture we will study the simplified equation

$$y_{tt} = -\alpha^2 y_{xxxx}.$$

2 Vibrations of a simply supported beam

We take the beam to have length $L = 1$ and be simply supported. This leads to the following IBVP:

*These notes were originally written by Menny Akka and edited by Martin Larsson. Some material was also taken from Alessandra Iozzi's notes.

Find $y = y(x,t)$ such that

$$(1)\qquad \begin{cases} y_{tt} = -\alpha^2 y_{xxxx} & \text{in } (0,1)\times(0,\infty), & \text{(PDE)} \\ y(0,t) = y_{xx}(0,t) = 0 & \multirow{2}{*}{for all $t>0$,} & \text{(BC)} \\ y(1,t) = y_{xx}(1,t) = 0 & & \\ y(x,0) = f(x) & \text{for all } x \in (0,1). & \text{(IC)} \\ y_t(x,0) = g(x) & & \end{cases}$$

Here $\alpha > 0$ is a given constant, and $f(x)$ and $g(x)$ are given functions.

To solve this IBVP we rely on familiar methods: separation of variables, superposition, and Fourier series. We start with the Ansatz

$$y(x,t) = X(x)\,(A\cos(\omega t) + B\sin(\omega t))$$

for some constants ω, A, and B that we would like to determine. Note that we took a shortcut here by directly making the Ansatz that the solution $y(x,t)$ should oscillate in t. We could also have started with the more general Ansatz $X(x)T(t)$.

Plugging the Ansatz into the PDE gives

$$X(x)\left(-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right) = -\alpha^2 X''''(x)\,(A\cos(\omega t) + B\sin(\omega t))\,.$$

Dividing both sides by $(A\cos(\omega t) + B\sin(\omega t))$ gives the ODE

$$X''''(x) - \frac{\omega^2}{\alpha^2}X(x) = 0.$$

Its general solution is

$$X(x) = C\cos\left(\sqrt{\frac{\omega}{\alpha}}\,x\right) + D\sin\left(\sqrt{\frac{\omega}{\alpha}}\,x\right) + E\cosh\left(\sqrt{\frac{\omega}{\alpha}}\,x\right) + F\sinh\left(\sqrt{\frac{\omega}{\alpha}}\,x\right)$$

Exercise 2.1. Derive this general solution using the Laplace transform!

In order to determine the constants we use the boundary conditions. In particular,

$$\begin{aligned} y(0,t) &= 0 &\implies&& X(0) &= 0 &\implies&& C + E &= 0, \\ y_{xx}(0,t) &= 0 &\implies&& X''(0) &= 0 &\implies&& -C + E &= 0, \end{aligned}$$

and from this we get $C = E = 0$. Taking this into account we use the remaining boundary conditions to obtain

$$\begin{aligned} y(1,t) &= 0 &\implies&& D\sin\left(\sqrt{\frac{\omega}{\alpha}}\right) + F\sinh\left(\sqrt{\frac{\omega}{\alpha}}\right) &= 0, \\ y_{xx}(1,t) &= 0 &\implies&& -D\sin\left(\sqrt{\frac{\omega}{\alpha}}\right) + F\sinh\left(\sqrt{\frac{\omega}{\alpha}}\right) &= 0. \end{aligned}$$

From this it follows that

$$D \sin\left(\sqrt{\frac{\omega}{\alpha}}\right) = 0 \qquad \text{and} \qquad F \sinh\left(\sqrt{\frac{\omega}{\alpha}}\right) = 0,$$

and since $\sinh(\sqrt{\omega/\alpha}) \neq 0$ we must have $F = 0$. We don't want $D = 0$ however, since this would result in the zero solution. Therefore $\sin(\sqrt{\omega/\alpha}) = 0$, so that

$$\frac{\omega}{\alpha} = (n\pi)^2, \qquad n = 1, 2, \ldots.$$

We end up with solutions of the form

$$y(x,t) = \sin(n\pi x)\left(A\cos(\alpha(n\pi)^2 t) + B\sin(\alpha(n\pi)^2 t)\right),$$

which satisfy (PDE) and (BC), but not yet (IC).

At this point we use superposition to get the more general solution

$$y(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(a_n\cos(\alpha(n\pi)^2 t) + b_n\sin(\alpha(n\pi)^2 t)\right),$$

where a_n and b_n are constants. By superposition, this solution still satisfies (PDE) and (BC). Its initial value and initial velocity are

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \tag{2.1}$$

$$y_t(x,0) = \sum_{n=1}^{\infty} b_n \alpha(n\pi)^2 \sin(n\pi x). \tag{2.2}$$

The initial conditions demand that these should match $f(x)$ and $g(x)$, respectively. By expressing $f(x)$ and $g(x)$ as Fourier sine series, we can achieve this by simply matching the coefficients. Therefore,

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)dx, \qquad b_n = \frac{2}{\alpha(n\pi)^2}\int_0^1 g(x)\sin(n\pi x)dx. \tag{2.3}$$

Let us summarize what we have found:

The general solution of IBVP (1) for vibrations of a simply supported beam of length $L = 1$ is given by

$$y(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(a_n\cos(\alpha(n\pi)^2 t) + b_n\sin(\alpha(n\pi)^2 t)\right),$$

where

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)dx,$$

$$b_n = \frac{2}{\alpha(n\pi)^2}\int_0^1 g(x)\sin(n\pi x)dx.$$

Exercise 2.2. Work out the general solution for a general length L, not necessarily $L = 1$.

Example 2.3. Let's take $\alpha = 1$, $y(x, 0) = \sin(\pi x) + 0.5 \sin(3\pi x)$, and $y_t(x, 0) = 0$. In this case it is easier to directly identify a_n and b_n from (2.1)–(2.2) than to compute the integrals in (2.3). Either method works however, and gives

$$\begin{aligned} b_n &= 0, \quad n \geq 1, \\ a_1 &= 1, \\ a_2 &= 0, \\ a_3 &= 0.5, \\ a_n &= 0, \quad n \geq 1. \end{aligned}$$

Therefore the solution is

$$y(x, t) = \sin(\pi x) \cos(\pi^2 t) + 0.5 \sin(3\pi x) \cos(9\pi^2 t).$$

It is interesting to compare this with the solution of the wave equation for vibrations of a finite string (with $L = 1$, $c = 1$) with the same initial condition, which is

$$u(x, t) = \sin(\pi x) \cos(\pi t) + 0.5 \sin(3\pi x) \cos(3\pi t).$$

You see that a beam vibrates at higher frequencies than a string. The frequencies of the string are multiples n of a basic frequency, while for the beam the multiples are n^2.

3 Solving PDE using the Laplace transform

We now leave the beam equation, and return to the Laplace transform. So far we have used the Laplace to solve ODE, but it can also be used to solve PDE. We finish this lecture with an example of how that works. The material of this section is from pages 101–103 of Farlow's book.

Let us consider heat flow in a semi-infinite medium; you should think of a pool which is deep enough that the boundary effects at the bottom are negligible. We approximate this situation by considering an infinitely deep pool. At the surface there are boundary effects, however: if the surface temperature $u(0, t)$ is larger than the air temperature (which we normalize to zero), then heat flows out. If $u(0, t)$ is less than the air temperature, then heat flows in. At time zero we assume that the pool has a constant uniform temperature of ϕ_0. This is modeled by the following IBVP:

Find $u = u(x, t)$ such that

$$(2) \qquad \begin{cases} u_t = u_{xx} & \text{in } (0, \infty) \times (0, \infty), & \text{(PDE)} \\ u_x(0, t) = u(0, t) & \text{for all } t > 0, & \text{(BC)} \\ u(x, 0) = \phi_0 & \text{for all } x > 0. & \text{(IC)} \end{cases}$$

Here ϕ_0 is a given constant.

The method we'll use to solve this IBVP is to take the Laplace transform in the t variable, but not in the x variable. That means considering the function

$$U(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)dt.$$

An important property of the Laplace transform is that we may interchange differentiation in x with the Laplace transform. In particular,

$$\mathcal{L}\left\{\frac{\partial^2}{\partial x^2} u(x,t)\right\} = \frac{\partial^2}{\partial x^2}\mathcal{L}\{u(x,t)\}.$$

Therefore, the PDE gives us

$$sU(s,x) - u(x,0) = \frac{\partial^2}{\partial x^2} U(x,s).$$

Taking into account the initial condition, our IBVP turns into the following problem:

$$\begin{cases} sU(x,s) - \phi_0 = \frac{\partial^2}{\partial x^2} U(x,s) & \text{in } (0,\infty)\times(0,\infty), \\ \frac{\partial}{\partial x} U(0,s) = U(0,s) & \text{for all } s > 0. \end{cases}$$

Think about what just happened: by taking the Laplace transform, we got rid of the derivative with respect to t. If we freeze the s variable, the above is nothing but an ODE in the x variable! To make this easier to see, let's define the function

$$V(x) = U(x,s).$$

It satisfies

$$sV(x) - \phi_0 = V''(x), \qquad V'(0) = V(0).$$

The general solution (homogeneous plus particular) of this ODE is

$$V(x) = Ae^{\sqrt{s}\,x} + Be^{-\sqrt{s}\,x} + \frac{\phi_0}{s}.$$

Since we don't want exploding temperatures, $u(x,t)$ should be bounded. Therefore $V(x) = U(x,s)$ should also be bounded, which means that A must be zero. The boundary condition $V'(0) = V(0)$ then tells us

$$-\sqrt{s}B = B + \frac{\phi_0}{s},$$

hence

$$B = -\phi_0 \frac{1}{s(\sqrt{s}+1)}.$$

Therefore

$$U(x,s) = V(x) = -\phi_0 \frac{e^{-\sqrt{s}\,x}}{s(\sqrt{s}+1)} + \frac{\phi_0}{s}$$

Finding the inverse transform $u(x,t) = \mathcal{L}^{-1}\{U(x,s)\}$ is a bit tricky, but can be done. The resulting solution is

$$u(x,t) = -\phi_0\left(\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\sqrt{t} + \frac{x}{2\sqrt{t}}\right)e^{x+t}\right) + \phi_0,$$

where $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2} d\xi$, and is called the **complementary-error function**.