Analysis III for D-BAUG, Fall 2017 — Lecture 5

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0 Recall from last time:

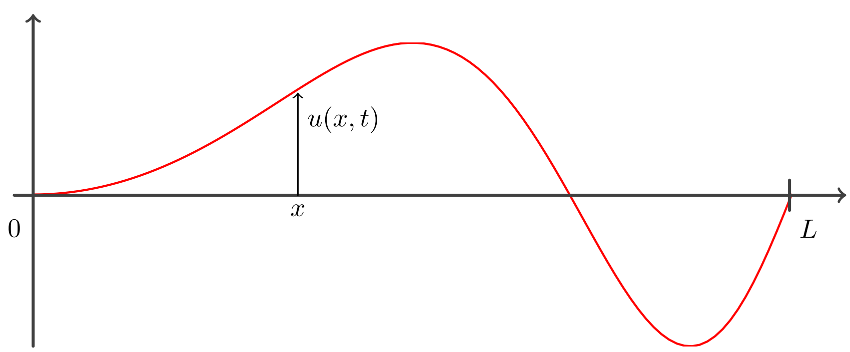
Last time we finally managed to solve our three IBVPs involving the one-dimensional heat equation with general initial conditions. Three key methods were used:

- **Separation of variables:** We used the Ansatz $u(x,t) = X(x)T(t)$ to find solutions for specific initial conditions (sines and cosines)
- **Superposition principle:** Taking linear combinations of those solutions allowed us to handle more complicated initial conditions (linear combinations of sines and cosines)
- **Fourier series:** Finally we used the theory of Fourier series to show that this actually covers the general case, if we allow for infinite linear combinations of sines and cosines.

We now leave the heat equation, and instead focus on the **one-dimensional wave equation**, which is the key example of a hyperbolic PDE. The above techniques again come in handy when solving certain IBVPs associated with the wave equation.

1 Vibrations of a finite string

Let $u(x,t)$ denote the vertical displacement of a string from its rest position at time t and horizontal location x.



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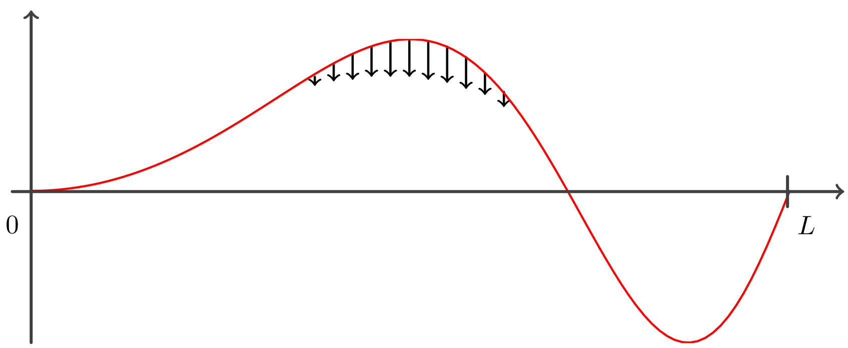
The wave equation describes how the string moves over time:

$$u_{tt} = c^2 u_{xx},$$

where $c > 0$ is a constant. The constant c turns out to have a meaning as the *speed of propagation* of the wave. We will discuss this later on.

For now, let's think about why this PDE makes sense as a model for the movement of a frictionless string. In fact it is a simplified model in which points on the string only move vertically. Hence, $u_{tt}(x_0, t_0)$ is the acceleration of the string at location x_0 and time t_0. By Newton's second law, this is proportional to the force acting on the string at that point. The wave equation therefore tells us that the force acting on the string at point x_0 and time t_0 is proportional to the second derivative $u_{xx}(x_0, t_0)$. This is a measure of the convexity of the string at that point:



In particular, if $u_{xx}(x_0, t_0)$ is *positive*, there is a net *upward* force acting on the string at the point x_0. If $u_{xx}(x_0, t_0)$ is *negative*, there is a net *downward* force acting on the string at the point x_0. The equation can be derived by thinking of the string as a chain of many weights connected by very short springs, and thinking about which direction each spring is pulling towards. If there is no change in direction (the case $u_{xx} = 0$) all forces act along the same line and they cancel out, while a non-zero second derivative gives rise to changes in direction and non-cancelling forces.

We will solve the following IBVP, which models the vibrations of a finite string of length L with fixed endpoints and given initial shape and velocity:

Find $u = u(x, t)$ such that

$$(1) \qquad \begin{cases} u_{tt} = c^2 u_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), & \text{(PDE)} \\ u(0, t) = u(L, t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ u(x, 0) = f(x) & & \\ u_t(x, 0) = g(x) & \text{for all } x \in (0, L). & \text{(IC)} \end{cases}$$

Here $c > 0$ and $L > 0$ are given constants, and $f(x)$ and $g(x)$ are given functions.

Here $f(x)$ represents the initial shape of the string, while $g(x)$ represents the initial velocity.

2 Solving the IBVP

Let us now solve the above IBVP using the three techniques we have developed so far: separation of variables, superposition principle, and Fourier series. In a first step, we simplify the problem by solving two separate IBVPs, one with zero initial shape $f(x)$, and one with zero initial velocity $g(x)$:

$$\text{(1a)} \quad \begin{cases} v_{tt} = c^2 v_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), & \text{(PDE)} \\ v(0, t) = v(L, t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ v(x, 0) = 0 & \\ v_t(x, 0) = g(x) & \text{for all } x \in (0, L). & \text{(IC)} \end{cases}$$

and

$$\text{(1b)} \quad \begin{cases} w_{tt} = c^2 w_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), & \text{(PDE)} \\ w(0, t) = w(L, t) = 0 & \text{for all } t > 0, & \text{(BC)} \\ w(x, 0) = f(x) & \\ w_t(x, 0) = 0 & \text{for all } x \in (0, L). & \text{(IC)} \end{cases}$$

Once we have a solution $v(x, t)$ of IBVP (a) and a solution $w(x, t)$ of IBVP (b), then the superposition principle implies that the function

$$u(x, t) = v(x, t) + w(x, t)$$

is a solution of the original IBVP (1).

Solving IBVP (a). We proceed as we did with the heat equation. Consider the Ansatz $v(x, t) = X(x)T(t)$ and insert this into the PDE to obtain

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Rearranging this gives

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)},$$

which must be equal to some constant K. Let's think about whether K should be positive or negative.

- $K = 0$: We obtain the equation $X''(x) = 0$, which implies $X(x) = A + Bx$ for some constants A and B. In order to match (BC), we must have $A = B = 0$. Thus we end up only with the zero solution.
- $K = \lambda^2 > 0$, where $\lambda > 0$: In this case we obtain the equation $X''(x) = \lambda^2 X(x)$. Its solution is

  $$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

  for some constants A and B. To match (BC) we need $A = -B$. Moreover, we need $X(L) = A \cdot (e^{\lambda L} - e^{-\lambda L}) = 0$, which gives $A = 0$. Again, we only obtain the zero solution.

- $K = -\lambda^2 < 0$, where $\lambda > 0$, is now the only remaining possibility, and the one that will yield interesting solutions. We obtain the two ODEs

$$X''(x) = -\lambda^2 X(x) \qquad \text{and} \qquad T''(t) = -c^2\lambda^2 T(t),$$

whose solutions are

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x) \qquad \text{and} \qquad T(t) = C\sin(c\lambda t) + D\cos(c\lambda t)$$

for some constants A, B, C, D. To match (BC) we must have $X(0) = X(L) = 0$ (unless $C = D = 0$, but this gives the zero solution), which gives $B = 0$ and $\lambda = \frac{n\pi}{L}$ for some positive integer n (or $A = 0$ and, again, the zero solution). To match the first (IC) we must have $T(0) = 0$, which gives $D = 0$.

In conclusion, we obtain

$$v(x,t) = AC\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi c}{L}t\right).$$

If we choose $AC = \frac{L}{n\pi c}$, we see that this function solves IBVP (a) with initial condition $g(x) = \sin\left(\frac{n\pi}{L}x\right)$. For general initial condition $g(x)$, we develop $g(x)$ as a Fourier sine series,

$$g(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right).$$

The superposition principle then implies that the solution of IBVP (a) with this initial condition is given by

$$v(x,t) = \sum_{n=1}^{\infty} c_n \frac{L}{n\pi c}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi c}{L}t\right).$$

Solving IBVP (b). This is very similar. Just as before, we use the Ansatz $w(x,t) = X(x)T(t)$. With the same reasoning using (BC) it follows that

$$X(x) = A\sin\left(\frac{n\pi}{L}x\right) \qquad \text{and} \qquad T(t) = C\sin\left(\frac{n\pi c}{L}t\right) + D\cos\left(\frac{n\pi c}{L}t\right)$$

for some constants A, C, D and some $n \in \mathbb{Z}$. Now, however, the (IC) is different. In particular, the second (IC) is $w_t(x,0) = 0$, which means that we need $T'(0) = 0$. Since $T'(0) = C \cdot \frac{n\pi c}{L}$, we see that $C = 0$. Therefore (with $A = D = 1$) we find that

$$w(x,t) = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right)$$

solves IBVP (b) with initial condition $f(x) = \sin\left(\frac{n\pi}{L}x\right)$. Developing $f(x)$ a Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right),$$

the superposition principle gives the corresponding solution of IBVP (b):

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right).$$

Combining the two solutions. One final application of the superposition principle lets us combine the solutions of IBVPs (a) and (b):

The general (physical) solution of IBVP (1) for vibrations of a finite string with fixed endpoints,

$$
\begin{cases}
u_{tt} = c^2 u_{xx} & \text{in } \Omega = (0, L) \times (0, \infty), \qquad \text{(PDE)} \\
u(0,t) = u(L,t) = 0 & \text{for all } t > 0, \qquad \text{(BC)} \\
u(x,0) = f(x) & \\
u_t(x,0) = g(x) & \text{for all } x \in (0, L). \qquad \text{(IC)}
\end{cases}
$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(b_n \cos\left(\frac{n\pi c}{L}t\right) + \frac{c_n L}{n\pi c}\sin\left(\frac{n\pi c}{L}t\right)\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$
$$c_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Remark 2.1. At first glance this might look similar to the solutions of the IBVPs associated with the heat equation. However, the solutions are very different. As a function of t, the solutions keeps oscillate like sine / cosine, without subsiding (remember that we have not taken friction into account!). The heat equation, on the other hand, exhibits exponential damping in time.

3 Interpretation of c

It was mentioned above that the constant c can be interpreted as the *speed of propagation*. This interpretation will become very clear later on when we consider vibrations of a string of infinite length. However, already in the context of IBVP (1) above, we can get an idea of why this interpretation is valid.

To make things simple, let $L = \pi$ and suppose we are interested in (IC) given by $f(x) = \sin(x)$ and $g(x) = 0$. From the previous section, the solution is then given by

$$u(x,t) = \sin(x)\sin(ct)$$

By a standard trigonometric identity (which we in fact used last time), this can be written as

$$u(x,t) = \frac{1}{2}\sin(x - ct) + \frac{1}{2}\sin(x + ct).$$

Now, for any function $h(x)$, the graph of the function $h(x-a)$ looks exactly like the graph of $h(x)$, just shifted to the right by a distance a. Therefore, $u(x,t)$ can be seen as the superposition of two fixed waveforms that move in opposite directions by a distance c per unit time t. In other words, $u(x,t)$ is superposition of waveforms that *propagate at constant speed* c. The figure below shows the graphs of $\sin(x)$ and $\sin(x-ct)$.