Analysis III for D-BAUG, Fall 2016 — Lecture 7

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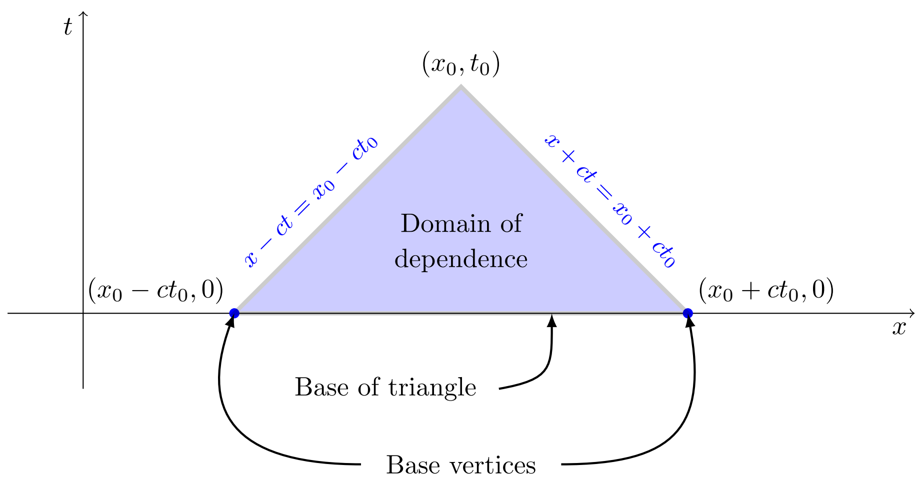
1 Characteristic lines

Last time we considered the one-dimensional wave equation,

$$u_{tt} = c^2 u_{xx}, \qquad (x,t) \in \mathbb{R} \times (0,\infty). \tag{1.1}$$

We ended by defining the **characteristic lines** of a point (x_0, t_0), and observing that they determine a triangle, which is known as the **characteristic triangle** or **domain of dependence**:



Recall D'Alembert's solution

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy,$$

where $f(x)$ and $g(x)$ specify the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. It follows from this expression that the solution $u(x_0,t_0)$ at the point (x_0,t_0) is completely determined by $f(x)$ and $g(x)$ on the base and the base vertices of the domain of dependence. In fact, the solution $u(x,t)$ at *any point* (x,t) *inside the domain of dependence* is pinned down by $f(x)$ and $g(x)$ on the base and the base vertices.

We also consider the **region of influence** of an interval $[x_1, x_2]$:

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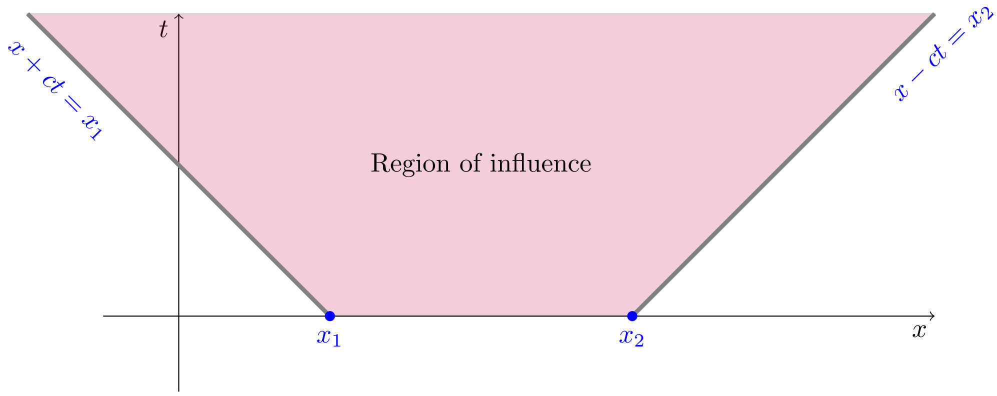

Its name comes from the fact that the solution $u(x, t)$ at points (x, t) inside the region of influence depend on (but are not necessarily fully determined by) the initial conditions $f(x)$ and $g(x)$ on the interval $[x_1, x_2]$.

Both the domain of dependence and the region of influence are bounded by two characteristic lines. In formulas, these lines are given by

$$x - ct = \text{const.} \qquad \text{and} \qquad x + ct = \text{const.}$$

For the domain of dependence of a point (x_0, t_0), the two constants are $x_0 - ct_0$ and $x_0 + ct_0$, respectively. For the region of influence of an interval $[x_1, x_2]$, the constants are x_1 and x_2, respectively. Notice the connection with the change of variables $\xi = x - ct$ and $\eta = x + ct$ that we used to derive D'Alembert's solution of the wave equation:

> Let $\xi(x, t) = x - ct$ and $\eta(x, t) = x + ct$ be the change of variables that transforms the wave equation (1.1) into the much simpler equation $u_{\xi\eta} = 0$. The characteristic lines are precisely those lines in the (x, t) plane along which either $\xi(x, t)$ or $\eta(x, t)$ is constant.

2 Method of characteristics

We mentioned last time that the change of variables technique used to derive D'Alembert's solution can be used for more general PDE. We now describe a general method to find the "right" change of variables. Consider the following PDE,

$$Au_{xx} + Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y), \tag{2.1}$$

where the coefficients A, B, and C *are allowed to depend on* x *and* y, and F on the right-hand side is some function. That is, we consider a second-order PDE with possibly non-constant coefficients. We do not worry about initial conditions, nor boundary conditions. The goal is to find new variables (ξ, η) such that the PDE becomes simpler in these new variables. Here is a recipe for doing this.

Method of characteristics.

(i) Start with a hyperbolic PDE (2.1).

(ii) Regard y as a function of x, and consider the following so-called **characteristic ODE**:

$$A(x,y)(y')^2 - B(x,y)y' + C(x,y) = 0,$$

where y' stands for $\frac{dy}{dx}$.

(iii) Find two independent solutions $y = y(x)$ of the characteristic ODE.[a]

(iv) Find a function $\xi(x,y)$ so that, when plugging in one of the solutions $y = y(x)$ from the previous step, you have $\xi(x, y(x)) = \text{const.}$.

(v) Find a function $\eta(x,y)$ that does the same for the other solution.

Then, using ξ and η as new variables, the PDE (2.1) takes the form

$$u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$$

for some function G.

[a]Two solutions are independent if at every x they have different derivatives.

Example 2.1. Let's verify that this works for the one-dimensional wave equation (1.1), writing y instead of t. Then $A = -c^2$, $B = 0$, and $C = 1$. The function F is just zero. The characteristic ODE becomes

$$-c^2(y')^2 + 1 = 0,$$

which just says that $y' = \pm\frac{1}{c}$. Thus, two independent solutions of the characteristic ODE are $t(x) = \frac{x}{c}$ and $t(x) = -\frac{x}{c}$. To find ξ, notice that $t = \frac{x}{c}$ can be written as $x - ct = 0$. Hence set $\xi(x,t) = x - ct$, and notice that by design $\xi(x, t(x)) = x - c\frac{x}{c} = 0$. Hence, ξ is one of the functions we are looking for. The other one is $\eta(x,t) = x + ct$, as can be easily verified.

Hence, the method of characteristics allows one to algorithmically find the correct "guess" for the change of variables we used to solve the wave equation by reducing it to the simplified PDE $u_{\xi\eta} = 0$.

If the PDE (2.1) has non-constant coefficients, the resulting change of variables will be nonlinear. We now consider an example of this situation.

Example 2.2. Consider the PDE

$$xu_{xx} - yu_{xy} + u_x = 0.^1 \tag{2.2}$$

For this PDE we have $A = x$, $B = -y$, $C = 0$. Therefore, $B^2 - 4AC = (-y)^2 > 0$, so the equation is hyperbolic. Its characteristic ODE is

$$x(y')^2 + yy' = 0.$$

[1]for $x > 0$.

This is equivalent to $(xy' + y)y' = 0$, which is equivalent to

$$y' = 0 \qquad \text{or} \qquad xy' + y = 0.$$

One solution is $y(x) = 0$, and we can just set $\xi(x, y) = y$. The other solution we can consider is $y(x) = 1/x$ (which is indeed independent of the solution $y(x) = 0$). Since $y = 1/x$ can be re-arranged as $xy = 1$, we see that we can choose $\eta(x, y) = xy$.

Using the chain rule (exercise!) we find that

$$xu_{xx} - yu_{xy} + u_x = y^2 u_{\xi\eta}.$$

Therefore the original PDE (2.2) holds if and only if

$$u_{\xi\eta} = 0.$$

We already know what the solution of this PDE is (we solved it last time): $u(\xi, \eta) = \Phi(\xi) + \Psi(\eta)$, where Φ and Ψ are two arbitrary functions. Switching back to the original variables (x, y), we obtain the general solution of the PDE (2.2):

$$u(x, t) = \Phi(y) + \Psi(xy).$$