Analysis III for D-BAUG, Fall 2017 — Lecture 8

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1 Transforms

The following picture from Farlow's book illustrates the basic idea of transforms in solving mathematical problems:

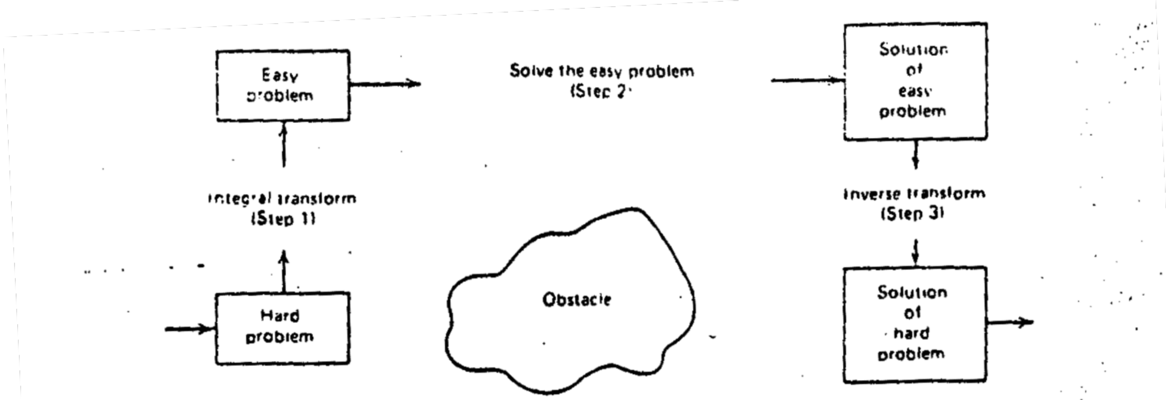

FIGURE 10.1 General philosophy of transforms.

To give a more concrete illustration, consider the following problem of multiplying two large numbers by using logarithms. Historically, the transform and inverse transform steps (taking logarithms and exponentials) were performed by looking up the numbers in tables. Adding the two logarithms is an easier and more reliable operation than directly multiplying the two original numbers, and this saved a lot of time in particular to astronomers before the advent of computers.

*These notes were originally written by Menny Akka and edited by Martin Larsson. Some material was also taken from Alessandra Iozzi's notes.

$$
\begin{tikzcd}
\log(932) + \log(587) \arrow[r, "\text{add}"] & 13.212358 \arrow[d, "\text{compute exponential}"] \\
932 \times 587 \arrow[u, "\text{compute logarithm}"] & 547084
\end{tikzcd}
$$

The Laplace transform uses the same philosophy. A typical application is to transform a differential equation (which may be tricky to solve directly) into an algebraic equation (which is solvable by simply re-arranging the terms), and then apply the inverse transform to obtain the solution of the original differential equation. Here is an example:

$$
\begin{tikzcd}
-1 + sF(s) \;=\; F(s) \arrow[r, "\text{solve the algebraic equation}"] & F(s) = \dfrac{1}{s-1} \arrow[d, "\text{inverse Laplace transform}"] \\
f'(t) = f(t), \quad f(0) = 1 \arrow[u, "\text{Laplace transform}"] & f(t) \;=\; e^t
\end{tikzcd}
$$

What is not revealed by this diagram is how to pass from the original equation $f'(t) = f(t)$ with initial condition $f(0)$, to the algebraic equation involving the Laplace transform $F(s)$, and then back from the solution $F(s)$ of the algebraic equation to the solution $f(t)$ of the original equation. That's our next task: to develop the theory of Laplace transforms, and use them to solve problems!

Finally, it should be mentioned that there are many other transforms that are sometimes useful.

2 The Laplace transform

Definition. The **Laplace transform** of a given function $f(t)$ is the function $F(s)$ given by

$$F(s) = \int_0^\infty e^{-st} f(t)dt. \tag{2.1}$$

We also denote the Laplace transform of $f(t)$ by $\mathcal{L}\{f(t)\}$ or $\mathcal{L}\{f(t)\}(s)$. The **inverse Laplace transform** of a given function $F(s)$ is the function $f(t)$ such that (2.1) holds. We also denote the inverse Laplace transform of $F(s)$ by $\mathcal{L}^{-1}\{F(s)\}$ or $\mathcal{L}^{-1}\{F(s)\}(t)$.

The obvious questions of when the Laplace transform and its inverse exist will be postponed for now. Instead, we'll first look at some examples. This involves computing indefinite integrals of the form $\int_0^\infty a(t)dt$ where $a(t)$ is some function. Recall how these are defined:

$$\int_0^\infty a(t)dt = A(t)\Big|_0^\infty = \lim_{T\to\infty} A(t)\Big|_0^T = \lim_{T\to\infty} A(T) - A(0)$$

whenever the limit exists, and where $A(t)$ is an antiderivative of $a(t)$: $A'(t) = a(t)$.

3 Examples of Laplace transforms

Example 3.1. We start with the simplest possible case: $f(t) = 1$ for all $t \geq 0$. Then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \lim_{T\to\infty} \frac{e^{-sT} - 1}{-s} = \frac{1}{s} \qquad \text{(for } s > 0\text{)}.$$

Notice that in this case, the Laplace transform $F(s) = \mathcal{L}\{1\}$ only exists for $s > 0$. As we will see, it is often the case that the Laplace transform only exists for sufficiently large values of s.

Example 3.2. Let $f(t) = e^{at}$ for some fixed constant a. Then

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = \frac{1}{s-a} \qquad \text{(for } s > a\text{)}.$$

Again, you see that the Laplace transform only exists for sufficiently large s.

Example 3.3. Now consider power functions $f(t) = t^n$, where $n \geq 1$ is an integer. We obtain

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty e^{-st}t^n dt \\
&= \frac{e^{-st}t^n}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} nt^{n-1}dt && \text{(integration by parts)} \\
&= 0 + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt && \text{(since } n \geq 1\text{)} \\
&= \frac{n}{s}\mathcal{L}\{t^{n-1}\} && \text{(definition of Laplace transform).}
\end{aligned}$$

Iterating the equality $\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$ leads to

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\} = \frac{n}{s} \times \frac{n-1}{s} \times \mathcal{L}\{t^{n-2}\} = \cdots = \frac{n}{s} \times \frac{n-1}{s} \times \cdots \times \frac{2}{s} \times \frac{1}{s}\mathcal{L}\{t^0\}.$$

Finally, using that $\mathcal{L}\{t^0\} = \mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$ by Example 3.1, we write this as

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},$$

for all $s > 0$.

4 Linearity

The Laplace transform is linear:

Theorem. The Laplace transform satisfies

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} \tag{4.1}$$

for any constants α and β. Similarly, the inverse Laplace transform satisfies

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}. \tag{4.2}$$

In particular,

$$\mathcal{L}\{\alpha f(t)\} = \alpha\mathcal{L}\{f(t)\} \qquad \text{and} \qquad \mathcal{L}^{-1}\{\alpha F(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\}$$

Let us verify that this theorem is correct. If $f(t)$ and $g(t)$ are functions and α and β constants, then

$$\begin{aligned}
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^\infty e^{-st}\left(\alpha f(t) + \beta g(t)\right) dt \\
&= \alpha \int_0^\infty e^{-st} f(t) dt + \beta \int_0^\infty e^{-st} g(t) dt \\
&= \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\},
\end{aligned}$$

using simply that integration is linear. We have thus derived (4.1). To derive (4.2), suppose we are given $F(s)$, $G(s)$, α, and β. Define the inverses $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$. Plugging this into (4.1), which we just checked is a valid equality, we get

$$\mathcal{L}\{\alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}\} = \alpha\mathcal{L}\{\mathcal{L}^{-1}\{F(s)\}\} + \beta\mathcal{L}\{\mathcal{L}^{-1}\{G(s)\}\} = \alpha F(s) + \beta G(s).$$

Taking the inverse Laplace transform of the left- and right-hand sides, we then get

$$\alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\}.$$

This is nothing but (4.2).

Example 4.1. Linearity is often useful for computing Laplace transforms of functions that consist of simpler building blocks. For example, let $f(t) = 2t^2 - 8$. Then by linearity,

$$\mathcal{L}\{2t^2 - 8\} = 2\mathcal{L}\{t^2\} - 8\mathcal{L}\{1\}.$$

But we already know that $\mathcal{L}\{t^2\} = 2/s^3$ and $\mathcal{L}\{1\} = 1/s$ (for $s > 0$), so

$$\mathcal{L}\{2t^2 - 8\} = 2 \times \frac{2!}{s^3} - 8 \times \frac{1}{s} = \frac{4 - 8s^2}{s^3}$$

for all $s > 0$.

Example 4.2. Here is another example: Let $f(t) = \cosh(at)$, where a is a constant. Since the hyperbolic cosine is given by $\cosh(a) = \frac{1}{2}(e^{at} + e^{-at})$, linearity along with the Laplace transform of e^{at} (which we already know) give

$$\mathcal{L}\{\cosh(at)\} = \mathcal{L}\{\frac{1}{2}(e^{at} + e^{-at})\} = \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\} = \frac{1}{2} \times \frac{1}{s-a} + \frac{1}{2} \times \frac{1}{s+a},$$

at least if $s > a$ and $s > -a$, or equivalently, $s > |a|$. Simplifying, we get (verify this!)

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2} \qquad \text{(for } s > |a|\text{)}.$$

Example 4.3. The two previous examples show how linearity can be used to compute the Laplace transform of functions consisting of simpler building blocks. But be careful! The Laplace is **not** multiplicative:

> **Warning:** $\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ in general.

A simple example that illustrates this is the following. We already know that $\mathcal{L}\{1\} = 1/s$ for $s > 0$. Therefore, $\mathcal{L}\{1\}\mathcal{L}\{1\} = 1/s^2$. But this means that $\mathcal{L}\{1 \cdot 1\} \neq \mathcal{L}\{1\}\mathcal{L}\{1\}$!

5 Existence and uniqueness

In several of the examples we've seen so far, $\mathcal{L}\{f(t)\}(s)$ only exists for some, but not all, values of s. This raises an important concern about the inverse Laplace transform: if $\mathcal{L}\{f(t)\}(s)$ only exists for some values of s, how can we be sure that $f(t)$ can be uniquely recovered from its Laplace transform? In other words, how can we be sure that the inverse Laplace transform exists? We will now develop general conditions on the function $f(t)$ that guarantee that its Laplace transform exists, at least for enough values of s to ensure that the inverse Laplace transform also exists.

> **Definition:** A function $f(t)$ is called of **exponential order (with constant c)** if
>
> $$|f(t)| \leq Me^{ct}$$
>
> for all $t \geq 0$ and for some constant M.

It's useful to have a simple way to check that a given function is of exponential order. The following lemma gives one way of doing this.

Lemma: If $f(t)$ is continuous and the limit $\lim_{t\to\infty} \frac{f(t)}{e^{ct}}$ exists and is finite for some constant c, then $f(t)$ is of exponential order with constant c.

To verify that the lemma is true, suppose we have a function $f(t)$ such that the given limit exists for some constant c. Let's call the limit $L = \lim_{t\to\infty} \frac{f(t)}{e^{ct}}$. From the definition of lim, there exists a (possibly large) constant T such that $\frac{|f(t)|}{e^{ct}} \leq |L| + 1$ for all $t \geq T$. Moreover, since $\frac{|f(t)|}{e^{ct}}$ is continuous, it achieves a maximum value on the interval $[0, T]$; let's call it $K = \max_{0\leq t\leq T} \frac{|f(t)|}{e^{ct}}$. Combining these two bounds gives $|f(t)| \leq e^{ct} \max\{|L| + 1, K\}$ for all $t \geq 0$. In other words, $f(t)$ is of exponential order with constant c, which is exactly what we wanted to verify.

Exercise 5.1. To practice using the lemma, check that any polynomial is of exponential order. It may be a good idea to do this in several steps:

- First use the lemma to check that any power function t^n is of exponential order.
- Then, use the lemma a second time to check that if $f(t)$ and $g(t)$ are of exponential order, and if α and β are constants, then $\alpha f(t) + \beta g(t)$ is also of exponential order.
- Finally, use the previous two facts to deduce that any polynomial is of exponential order.

Once you have solved this exercise, you could also think about what can be said about the constant c in the definition of exponential order for the case of polynomials!

Theorem (existence). Let $f(t)$ be a continuous function that is of exponential order with constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ exists and is finite for all $s > c$.

To derive this result we use the following fact from calculus: If $a(t)$ is a continuous function with $\int_0^\infty |a(t)|dt < \infty$, then $\int_0^\infty a(t)dt$ exists and is also finite. Equipped with this fact, it's enough to check that $\int_0^\infty e^{-st}|f(t)|dt < \infty$ for any $s > c$. Since $f(t)$ is of exponential order (and using the comparison theorem for integrals), we get

$$\int_0^\infty e^{-st}|f(t)|dt \leq \int_0^\infty e^{-st} M e^{ct} dt = \frac{M}{s-c} < \infty$$

for any $s > c$. This is what we wanted to show.

Next, let's give a uniqueness result. It shows that it's enough that $\mathcal{L}\{f(t)\}(s)$ exists for all sufficiently large s in order to recover $f(t)$. Due to the existence theorem, this happens if $f(t)$ is of exponential order.

Theorem (uniqueness). Let $f(t)$ and $g(t)$ be continuous functions of exponential order. If there is a constant c such that $\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g(t)\}(s)$ for all $s > c$, then $f(t) = g(t)$ for all $t \geq 0$.