Analysis III for D-BAUG, Fall 2017 — Lecture 9

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Last time we introduced the Laplace transform of a function $f(t)$,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

We also studied **existence**, **uniqueness**, **linearity**, and considered a number of **examples**. Note that we will not compute the Laplace transform of all relevant functions in class. Instead we will often rely on the ready-made table available on the course website. Today we will look at further properties of the Laplace transform and its inverse, and use them to solve ODEs.

1 How to use $\mathcal{L}^{-1}$

It should be clear how to use the table to find inverse transforms:

Example 1.1. Let $F(s) = \frac{1}{1+s}$. Then, since $\mathcal{L}\{e^{-t}\} = \frac{1}{1+s}$, we have $\mathcal{L}^{-1}\{F(s)\} = e^{-t}$.

It often happens that a given function $F(s)$ does not directly appear in the table. Often one has to rewrite $F(s)$ and express it as a combination of functions which *do* appear in the table. One very useful technique for doing so is to use partial fractions:

Example 1.2. Let $F(s) = \frac{s^2+s+1}{s^3+s}$. Let's decompose $F(s)$ using partial fractions. The denominator can be factored as

$$s^3 + s = s(s^2 + 1).$$

Therefore we'd like to find constants A, B, C such that

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

Putting the terms on the right-hand side on a common denominator, we obtain the equation $s^2 + s + 1 = (A + B)s^2 + Cs + A$. Thus $A = 1$, $C = 1$, and $B = 0$, and we arrive at

$$F(s) = \frac{1}{s} + \frac{1}{s^2 + 1}.$$

Using linearity of the inverse Laplace transform along with the tables, we finally obtain

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = 1 + \sin(t).$$

*These notes were originally written by Menny Akka and edited by Martin Larsson. Some material was also taken from Alessandra Iozzi's notes.

A useful property of the Laplace transform is the following:

Lemma. The Laplace transform satisfies the s-*shifting* property:

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a) \qquad (\text{equivalently } e^{-at}f(t) = \mathcal{L}^{-1}\{F(s+a)\}),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

This lemma is easy to verify; simply use the definition of the Laplace transform to compute

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-st}e^{-at}f(t)dt = \int_0^\infty e^{-t(s+a)}f(t)dt = F(s+a).$$

Let's look at an example of how to use the s-shifting property.

Example 1.3. We'd like to find the inverse Laplace transform of $G(s) = \frac{1}{s^2+4s+8}$. By completing the square we find that

$$G(s) = \frac{1}{(s+2)^2+4} = F(s+2), \qquad \text{where } F(s) = \frac{1}{s^2+4}.$$

From the table and linearity of the inverse Laplace transform,

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin(2t).$$

The s-shifting property then finally gives

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{F(s+2)\} = e^{-2t}\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-2t}\sin(2t).$$

2 The Laplace transform of derivatives

In order to solve ODEs using the Laplace transform, we need to know what happens when we transform the derivative of a function $f(t)$. That is, we need to know how to compute $\mathcal{L}\{f'(t)\}$. Suppose $f(t)$ is of exponential order with constant c (this was introduced last time). Then, using the definition of the Laplace transform and integration by parts,

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st}f'(t)dt \\
&= \underbrace{e^{-st}f(t)\Big|_0^\infty}_{-f(0)\text{ for } s>c} - \int_0^\infty -se^{-st}f(t)dt \\
&= -f(0) + s\int_0^\infty e^{-st}f(t)dt \\
&= -f(0) + s\mathcal{L}\{f(t)\}, \qquad s > c.
\end{aligned}$$

In this way, the Laplace transform of the derivative $f'(t)$ is expressed in terms of the Laplace transform of the function $f(t)$ itself. This is a very powerful fact which makes the Laplace transform extremely useful for solving ODEs.

Let's look at $\mathcal{L}\{g''(t)\}$, where $g(t)$ and $g'(t)$ are of exponential order. If we let $f(t) = g'(t)$, we can use the above formula twice to get

$$\mathcal{L}\{g''(t)\} = \mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\} = -g'(0) + s\mathcal{L}\{g'(t)\} = -g'(0) - sg(0) + s^2\mathcal{L}\{g(t)\}.$$

In general, by repeating this n times, we obtain

Theorem. The Laplace transform of the nth derivative $f^{(n)}(t)$ of $f(t)$ is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0),$$

assuming that $f, f', f'', \dots, f^{(n)}$ are continuous and of exponential order.

The theorem also works under more general conditions, for example $f^{(n)}$ can be piecewise continuous, but we will not worry about the exact conditions in applications.

Example 2.1. Let's solve the ODE $f'(t) = f(t)$, $f(0) = 1$. Write $F(s) = \mathcal{L}\{f(t)\}$ and use the differentiation rule together with the condition $f(0) = 1$ to obtain

$$sF(s) - 1 = F(s).$$

This is an algebraic equation which is trivial to solve:

$$F(s) = \frac{1}{s-1}.$$

Now take the inverse Laplace transform to get

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t,$$

which is indeed the solution of the ODE. Of course, in this case you already knew this, but it turns out that the same elegant method works for a large class of ODEs which you may not know how to solve easily by other means.

Example 2.2. Consider now the slightly more complicated second-order ODE

$$\begin{aligned} f''(t) + f(t) &= \cos(2t) \\ f(0) &= 0 \\ f'(0) &= 1. \end{aligned}$$

Taking the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ and using the boundary conditions gives

$$-1 + s^2F(s) + F(s) = \mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4},$$

where the Laplace transform of $\cos(2t)$ is taken from the table. Again we have an algebraic equation whose solution is

$$F(s) = \frac{1}{s^2+1} + \frac{s}{(s^2+1)(s^2+4)}.$$

The final step is to compute the inverse transform. This is usually the most challenging part, and practice is needed in order to know how to do this in concrete examples. In this case the technique using partial fractions works well. As an exercise, use partial fractions to derive

$$F(s) = \frac{1}{s^2+1} + \frac{s/3}{s^2+1} - \frac{s/3}{s^2+4}.$$

From here it is now easy to use linearity of the inverse Laplace transform together with the table to get

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \\ &= \sin(t) + \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t). \end{aligned}$$

3 The Heaviside function

A very useful function in engineering is the *Heaviside function*:

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

This function is useful because it captures switching a signal on or off. Here are some examples of how this is used.

Example 3.1. (i) "Switch on a signal at time $t = a$": $f(t) = u(t-a)$.

(ii) "Switch on a signal at time $t = a$ and switch it off at $t = b > a$":

$$f(t) = u(t-a) - u(t-b) = \begin{cases} 0, & t < a, \\ 1, & a \leq t < b, \\ 0, & b \leq t. \end{cases}$$

(iii) "Switch on a sine signal at time $t = a$":

$$f(t) = \sin(t-a)u(t-a) = \begin{cases} 0, & t < a, \\ \sin(t-a), & t \geq a. \end{cases}$$

(iv) "Connect an already running sine signal at time $t = a$":

$$f(t) = \sin(t)u(t-a) = \begin{cases} 0, & t < a, \\ \sin(t), & t \geq a. \end{cases}$$

A more sophisticated use of the Heaviside function is to construct unusual functions by pasting together (or switching between) some given set of simple functions.

Example 3.2. Let $f_1(t), f_2(t), \ldots, f_n(t)$ be some functions, and let $0 \le a_0 < a_1 < \cdots < a_n$ be some constants. The function

$$f(t) = \sum_{j=1}^{n} \Big(u(t-a_{j-1}) - u(t-a_j) \Big) f_j(t)$$

is then equal to $f_j(t)$ whenever t lies in the interval $(a_{j-1}, a_j]$. To see that this is really so, have a look back at Example 3.1(ii).

Given this great flexibility of the Heaviside function for building customized functions, it is a lucky fact that it interacts very well with the Laplace transform. The following theorem gives the exact connection.

Theorem. The Laplace transform satisfies the *t-shifting property*, which we write in three different ways:

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

... or equivalently: $$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f(t)\}\}$$

... and also: $$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

Exercise 3.3. Verify the equality

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

in two different ways: (1) using the t-shifting theorem; and (2) directly from the definition of the Laplace transform.

Example 3.4. Recall that a mass-spring system with forcing $g(t)$ and zero initial displacement and velocity can be described mathematically as an ODE

$$\begin{aligned} f''(t) + f(t) &= g(t) \\ f(0) &= 0 \\ f'(0) &= 0. \end{aligned}$$

This is similar to the ODE we looked at in Example 2.2. Let's assume that the forcing is switched on at $t = 1$ and off at $t = 5$, that is, we suppose that

$$g(t) = u(t-1) - u(t-5).$$

Then, taking the Laplace transform and using the boundary conditions, we obtain the algebraic equation

$$s^2F(s) + F(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

Solving this equation gives

$$F(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

As an exercise, use partial fractions to show that $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos(t)$. Therefore, using the t-shifting theorem,

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-s}\mathcal{L}\{1-\cos(t)\}\} = (1-\cos(t-1))u(t-1).$$

In the same way we find $\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = (1-\cos(t-5))u(t-5)$, and combining these two equations finally gives

$$f(t) = (1-\cos(t-1))u(t-1) - (1-\cos(t-5))u(t-5).$$