Eidgenössische Technische Hochschule Zürich
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Analysis III

Lecture Notes 2017

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Disclaimer

Despite our effort to make these lecture notes coherent and error-free, typos and even mistakes do slip in. We would greatly appreciate your help in finding them.
Special thanks go to my PhD student Alexander Smirnow for his great support in preparing this material.
Please send your improvement suggestions to Alexander Smirnow at alexander.smirnow@bf.uzh.ch.

0 Introduction and Preliminaries

This first section is a brief summary of mathematical conventions and some 'well-known' results. Please refer to this section first if you have trouble with the notation. It is intended to get everyone on (roughly) the same page.
We fix a sort of standard in regard to how we communicate mathematical subjects. After all, being rigorous in your thoughts and precise in your notation is half the maths. Most of the concepts introduced here are fundamental and can be found in most of the introductory literature on applied mathematics. For further material we suggest '*Mathematik für Ingenieure und Naturwissenschaftler Band 2: Ein Lehr- und Arbeitsbuch für das Grundstudium*' by Lothar Papula.

0.1 Sets and quantifiers

In mathematics, we use a lot of symbols as a tool of presentation and abbreviation. Often the meaning cannot be deduced from the symbol itself, thus it is always good to explain your notation. Some symbols however are very strongly associated with a particular meaning, and changing them would result in confusion. Let us start with two basic symbols you should already be familiar with:

- The symbol $\in$ means 'is element of' or simply 'in'. So $x \in \mathbb{R}$ means 'the element x belongs to the set $\mathbb{R}$' and is usually pronounced 'x in $\mathbb{R}$'. If we say 'take $x \in \mathbb{R}$' we mean 'take the element x in the set $\mathbb{R}$.
- The symbol $\subset$ means 'is subset of'. If we write 'consider $A \subset \mathbb{R}$' we mean 'consider the subset A of $\mathbb{R}$'. In this case, $\mathbb{R}$ is the superset of A.

We will use some conventional sets of numbers. Since we use these sets all the time, we agree on certain symbols to denote these sets. Throughout this lecture, we will use the following notation.

- $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of *natural numbers*, or *naturals*.[1]

Note that $\mathbb{N}$ does not contain 0. If we include 0, we use the set

- $\mathbb{N}_0 = \{0\} \sqcup \mathbb{N}$, the disjoint union of 0 and the natural numbers.[2]

$\mathbb{N}$ and $\mathbb{N}_0$ come in handy when we count something. A set with the 'same size' as $\mathbb{N}$ is called *countably infinite*.[3]

Of course we are also interested in negative numbers. So a natural step would be to 'mirror' the naturals onto the 'negative side':

[1] You might wonder what comes after '3'. The correct answer is '4' and then '5'. After any $n \in \mathbb{N}$ there comes $n+1$. A mathematician might not be satisfies yet, but for us this definition is sufficient.

[2] Putting curly brackets around the 0 means that the element 0 is considered as a one-element set $\{0\}$.

[3] To be more precise, if there is a bijection from the set to $\mathbb{N}$.

- $\mathbb{Z} = \mathbb{N}_0 \sqcup \{-n \,|\, n \in \mathbb{N}\} = \mathbb{N}_0 \sqcup -\mathbb{N}$, the set of all *integer numbers* or *integers*.

Note that we may refer to $\mathbb{N}$ as the set of *all positive integers*, and $\mathbb{N}_0$ as the set of *all non-negative integers*. In general, 'positive' means '*strictly* positive' and 'non-negative' includes the 0. The symbol $-\mathbb{N}$ is an abbreviation for $\{-n \,|\, n \in \mathbb{N}\}$ and means that each element in $\mathbb{N}$ is multiplied by -1. More generally, we define for a set X the set $cX = \{cx \,|\, x \in X\}$. Elements from $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ are often denoted by the symbols j, k, ℓ, n, m. Sometimes, especially in programming, the letter i denotes a running variable in $\mathbb{N}$. We will try and avoid this, since i will denote the complex unit.

The next set contains all the fractions,

- $\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z}, q \in \mathbb{N} \right\}$, the set of *rational numbers*, or *rationals*.

We often use symbols q, r to denote elements of $\mathbb{Q}$.

The following set is a little bit more difficult to define in a rigorous but concise manor without many prerequisites. For our purposes we are satisfied with the following definition,

- $\mathbb{R} = \{\text{irrational numbers}\} \sqcup \mathbb{Q}$, the set of *real numbers*, or *reals*.

Irrational numbers are numbers that cannot be written as a fraction. So in particular, $\sqrt{2}, \pi, e$ and so on are contained in $\mathbb{R}$ (but not in $\mathbb{Q}$). Since $\mathbb{R}$ does not have any 'gaps'[4] we can conveniently use the interval notation. Intervals are special subsets of $\mathbb{R}$, and the numbers $a, b \in \mathbb{R}$ with $a \leq b$ are the *boundaries* of the intervals:

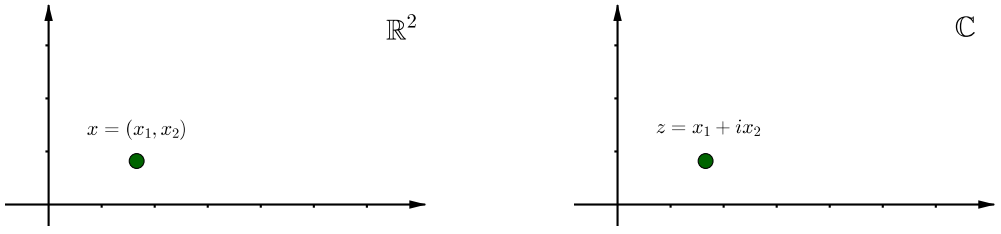
- $[a,b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$ is called a *closed interval*,
- $(a,b) = \{x \in \mathbb{R} \,|\, a < x < b\}$ is called an *open interval*,

The half-open intervals $(a,b]$ and $[a,b)$ are analogously defined. Furthermore, we will sometimes use the notation $\mathbb{R} = (-\infty, \infty)$ and also $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\} = [0, \infty)$, $\mathbb{R}_{>0} = \{x \in \mathbb{R} \,|\, x > 0\} = (0, \infty)$ and so on. We will usually use the symbols x, y, s, t to denote real numbers.

The last set we consider here, is the set of complex numbers. For this we introduce the imaginary unit $i = \sqrt{-1}$ and finish this brief recapitulation with

- $\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\}$, the set of all *complex numbers*.

We will often use z, w to denote complex numbers. As you might know, we can think of $\mathbb{C}$ as the product space $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Look at the following figure and find an argument why this is justified.



(1) Visualisation of the plane $\mathbb{R}^2$ (2) Visualisation of the 'plane' $\mathbb{C}$

Figure 0.1: Visualisation of $\mathbb{R}^2$ and $\mathbb{C}$ as planes

The real and the imaginary part of a complex number $z = x_1 + ix_2 \in \mathbb{C}$ are denoted by $\Re z = \mathrm{Re}(z) = x_1$ and $\Im z = \mathrm{Im}(z) = x_2$, respectively.

[4] Imagine that we filled all uncountably-infinite gaps in $\mathbb{Q}$ with irrational numbers.

We end this subsection with the introduction of *quantifiers*. These are symbols that allow us to abbreviate certain statements. However, these symbols **are *not*** a substitute for clear, explaining sentences and they **should *not*** be used outside formulae.

Let $A(x)$ be a statement about some $x \in \mathbb{R}$.

- The symbol $\forall$ stands for 'for all'. So if we write $\forall x \in \mathbb{R} : A(x)$, it means 'for all $x \in \mathbb{R}$ the statement $A(x)$ holds'.
- The symbol $\exists$ stands for 'there exists'. Writing $\exists x \in \mathbb{R} : A(x)$ is understood as 'there exists (at least one) $x \in \mathbb{R}$ such that the statement $A(x)$ holds'.
- We use $\exists!$ if we mean 'there exists *exactly one*'.
- The symbol $\nexists$ stands for 'there does not exist any'. So $\nexists x \in \mathbb{R} : A(x)$ means 'there is no $x \in \mathbb{R}$ such that $A(x)$ is satisfied'.

These symbols may seem very handy, and in fact, they sometimes are. Be careful though, the order of use is important, since $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : A(x,y)$ is, in general, not the same as $\exists y \in \mathbb{R}\ \forall x \in R : A(x,y)$. For example[5], let X be the set of students at the ETH and let Y be the set of all offered lectures. For $x \in X$ and $y \in Y$ let $A(x,y)$ be the statement: 'student x is interested in lecture y'. Then '$\forall x \in X\ \exists y \in Y : A(x,y)$' is (hopefully) true. But '$\exists y \in Y\ \forall x \in X : A(x,y)$' is (most likely) not true. What do you think?

0.2 Sequences, sums, and series

A collection or family of enumerated objects is called a *sequence*. The objects can be anything: numbers, vectors, functions, letters, houses on a particular road, results of coin tosses, and so on. The *length* of a sequence is defined by the number of elements in that sequence and it can be finite or infinite. Usually, we use $\mathbb{N}$ (countably infinite) or a subset $\{1,\dots,N\} \subset \mathbb{N}$ (finite) which is an abbreviation for $\{n \in \mathbb{N} \,|\, N \in \mathbb{N}, n \leq N\}$ to enumerate a sequence. We write

$$(a_n)_{n\in\mathbb{N}} \text{ to denote the sequence } (a_1, a_2, a_3, ...),$$

where a_n could be numbers, vectors, and so on, you know that already. In more generality, one can use an *index set I* instead of $\mathbb{N}$ to write $(a_n)_{n\in I}$. You can also write $(a_n)_{n=1}^{N}$ if your indeces are in the set $\{1,\dots,N\} \subset \mathbb{N}$.

Exercise 0.1 *For example, you might know the Fibonacci sequence,*

$$(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...).$$

Define $a_1 = 0$, $a_2 = 1$ and then find a recursive *rule to define the whole sequence. This means define any a_n for $n \geq 3$ using the previous terms of the sequence. In this case, you only need the two previous terms.*

Note, a sequence of elements from a set X is a subset of X. So for example, the Fibonacci sequence $(a_n)_{n\in\mathbb{N}}$ is a subset of $\mathbb{N}$, since each $a_n \in \mathbb{N}$. If we want to emphasise that the sequence should be considered as a subset, we sometimes use curly brackets $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$. Of course this is not true for elements of different sets, just as you cannot compare apples and oranges (except maybe if you chose 'fruits' as your set).

A sequence of functions $(f_n)_{n\in\mathbb{N}}$ from a set X to $\mathbb{R}$ is said to *converge pointwise*, if for each $x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

[5] Taken from 'Analysis I und II (2016/2017)', Manfred Einsiedler, Andreas Wieser, Beispiel 1.8.

With this notion we consider the limit for each x separately. In particular, we have no control over how 'fast' the functions converge for a given x. To consider all $x \in X$ at the same time, we introduce the notion of *uniform convergence*,

$$\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0\,.$$

Here, we are interested not only in the difference of one certain value x, but in the largest such difference of all $x \in X$. It is clear that uniform convergence implies pointwise convergence. The other direction is not true in general. A classic example is the following.

Example 0.2 Consider for $n \in \mathbb{N}$ the sequence of functions $f_n : [0,1] \to [0,1]$ defined by $x \mapsto x^n$. Note that the sequence converges pointwise to the function

$$f : [0,1] \to [0,1] \quad \text{with} \quad x \mapsto \begin{cases} 0 & \text{if } x \in [0,1)\,, \\ 1 & \text{if } x = 1\,. \end{cases}$$

Indeed, for $x = 1$ this is true, since $1^n = 1$ for all $n \in \mathbb{N}$. For $x \in [0,1)$ and a given bound $\epsilon > 0$ we can choose $n \geq N(\epsilon, x) = \left\lceil \frac{\log \epsilon}{\log x} \right\rceil$ to ensure that $|f_n(x) - f(x)| = |x^n| = x^n < \epsilon$. Hence, the bound $N(x,\epsilon)$ depends on ϵ and x.
To show that the sequence does not converge uniformly, we have to show that there is no such value $N(\epsilon)$ such that it converges for all $x \in [0,1]$ at the same time. And indeed, for any given N and ϵ we simply choose $x > \epsilon^{\frac{1}{N}}$ and we get $|f_n(x) - f(x)| > \epsilon$.

When we want to add countably many elements together we use the symbol $\sum$ (capital sigma). The summation of finitely many quantities is called a *sum* and we use running variables from the finite set $\{1,\dots,N\}$,

$$\sum_{n\in\{1,\dots,N\}} a_n = \sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N\,.$$

Summation over infinitely many quantities is called a *series* and we usually use elements in $\mathbb{N}$ as running variables,

$$\sum_{n\in\mathbb{N}} a_n = \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \ldots\,.$$

One has to understand that this notation does not guaranty that there is actually a value for this 'thing'. However, if the limit exists, we say the series is *convergent* or *summable* and we call the limit the *sum of the series*. If it does not, we can sometimes assign values like $+\infty, -\infty$, but other times it is just not possible, as for example $\sum_{n=0}^{\infty}(-1)^n$.[6]
A series is called *absolutely summable* if the series of the absolute values of its summands is summable.

Exercise 0.3 *1. Show the Gauß summation formula (*kleiner Gauß*)*

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}\,.$$

2. Furthermore derive

$$\sum_{n=1}^{N} 2n = N(N+1) \quad \textit{and} \quad \sum_{n=1}^{N} 2n - 1 = N^2\,.$$

[6] Maybe we should set it equal to $\frac{1}{2}$, what do you think? https://www.youtube.com/watch?v=PCu_BNNI5x4

3. Show that the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

converges to $\log 2$*. Explain why it is not absolutely convergent.*

0.3 Functions and graphs

Let us now focus our attention on functions. Functions are a key part of a lot of mathematics. The basic principle is quite simple and we will first explain it in words:

- We say f is a *function* (or *map*) from a set X to a set Y and we write $f : X \to Y$ if f is an assignment rule, which assigns to each element in X a unique element $y = f(x) \in Y$.

The set X, on which the function is defined, is called *domain* and the set Y is called *co-domain*[7]. Note that a map does not necessarily lead to all the elements in the co-domain. We can however restrict the co-domain to only those values $y \in Y$ such that there is an $x \in X$ with $f(x) = y$.

- The *image* of the function $f : X \to Y$ is denoted by $f(X)$ and defined as

$$f(X) = \{y \in Y \,|\, \exists x \in X : y = f(x)\} \subseteq Y\,.$$

Do not get confused by this notation: the function f is only defined on elements $x \in X$, it does not take sets as arguments. The notation $f(X)$ is just an abbreviation for the *set* we defined above.

Finally, we introduce the restriction of a function to a subset of its domain.

- The *restriction* of a function $f : X \to Y$ to a subset $A \subset X$ is denoted by $f|_A : A \to Y$ and defined point-wise as $f|_A(x) = f(x)$, for $x \in A$.

Important Do *not* think of functions just as formulae, as for example $y = x^2$. Keep the context of a domain and a co-domain in mind! If you define a function, then

1. give it a name, for example f,
2. choose a domain and a co-domain and indicate your choice as $f : X \to Y$,
3. finally, indicate that x is mapped to $f(x)$, or $x \mapsto f(x)$, where $f(x)$ can be an explicit formula.

Note that if we want to indicate that elements are mapped, we use the arrow $\mapsto$, and only between sets we use $\to$. For example, $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is a correct definition of '$y = x^2$'. In particular, the notation '$y = x^2$' might tempt you to think of the function as its graph. In fact, this is sometimes helpful, but we need to be careful to differentiate between a function and its graph.

- The graph Γ of a function $f : \mathbb{R} \to \mathbb{R}$ is a subset of $\mathbb{R}^2$. It is given by

$$\Gamma = \{(x, f(x)) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\} = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, f(x_1) = x_2\} \subset \mathbb{R}^2\,.$$

We deliberately use the notation (x_1, x_2) rather than (x, y), since it is easier to generalise. For example, $x = (x_1, \ldots x_n) \in \mathbb{R}^n$ is simpler than coming up with letters or symbols $x = (a, b, \ldots, z, \odot, \star, \text{☺})$.

Another important note $f : X \to Y$ is a function, but $f(x)$ is not, $f(x)$ is the function f evaluated at the point x and thus, it is just an element (or value) in Y. ***Do not*** call $f(x)$ a function, since it is not!

If you do not see the importance of stating the domain and co-domain, consider the following short exercise, in which you will find that two seemingly equal functions actually differ quit a lot, for example with regard to the existence of an inverse.

[7] In German we say 'Definitionsbereich' and 'Wertebereich', respectively.

Exercise 0.4 *Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $x \mapsto x^2$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $x \mapsto x^2$ be two functions. You might know that the square-root function $\sqrt{\cdot} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the inverse of g. Explain why it is* not *the inverse of f. Argue that f does not have an inverse. What about $f|_{\mathbb{R}_{\geq 0}}$?*

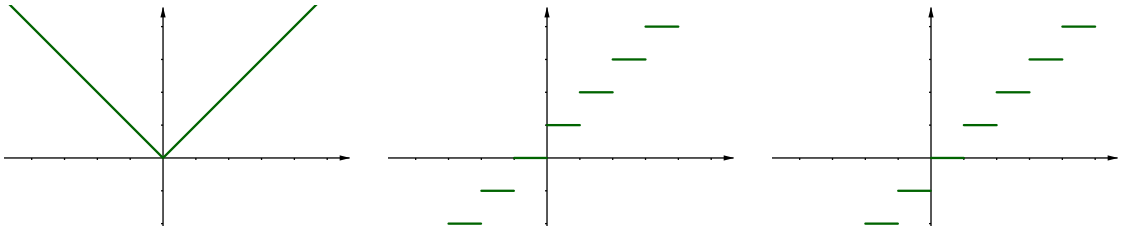
Example 0.5 Let us introduce some useful functions.

1. Let $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be defined by $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. The number $|x|$ is called the *absolute value* of x.

2. We call $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ defined as $\lceil x \rceil = \min\{n \in \mathbb{Z} \,|\, x \leq n\}$ the *ceiling function*. For any $x \in \mathbb{R}$ it gives you the smallest integer greater than or equal to x.
Similarly, we call $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ defined as $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \,|\, x \geq n\}$ the *floor function*. It gives the largest integer less than or equal to x.

3. We will often talk about the length of an interval $[a,b] \subset \mathbb{R}$. The length can be considered as a function λ that maps subsets of $\mathbb{R}$ to $\mathbb{R}$. Let $R = \{[a,b] \subset \mathbb{R} \,|\, a,b \in \mathbb{R}, a \leq b\}$ be the set of all closed, non-empty intervals. Then $\lambda : R \to \mathbb{R}$ is defined as $[a,b] \mapsto b-a$. In fact, λ can be extended to open and half-open intervals and we set $\lambda([a,b)) = \lambda((a,b]) = \lambda((a,b)) = \lambda([a,b]) = b-a$. In particular, the length of a point $x \in \mathbb{R}$ is equal 0. Furthermore, we set λ equal 0 for all empty intervals and ∞ for all intervals of the forms $(a,\infty), (-\infty,a)$ for $a \in \mathbb{R}$.

Let us look at the graphs of these functions in Figure 0.2.



(1) Graph of $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$ (2) Graph of $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ (3) Graph of $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$

Figure 0.2: Visualisation of the absolute value, the ceiling and floor functions.

In previous courses you have learnt how to integrate a real-valued function. We can interpret the integral of a function $\mathbb{R} \to \mathbb{R}$ as the area enclosed in between the graph of the function and the x-axis. We will not go into more detail here. Please make sure you feel comfortable with this notion and that you remember basic properties, such as linearity.
We define the two following types of integrability:

- A function $f : \mathbb{R} \to \mathbb{R}$ is called *integrable* if the integral over the domain is finite,
$$\left| \int_{\mathbb{R}} f(x)\,\mathrm{d}x \right| < \infty\,.$$
- A function is *absolutely integrable* if its absolute value is integrable,
$$\int_{\mathbb{R}} |f(x)|\,\mathrm{d}x \; < \infty\,.$$
If we define $f^+(x) = \max(f(x),0)$ and $f^-(x) = \max(-f(x),0)$ we can write $|f(x)| = f^+(x) - f^-(x)$. Using *Lebesgue integration*, we say f is integrable if $\int f^+(x)\,\mathrm{d}x - \int f^-(x)\,\mathrm{d}x < \infty$, so essentially, absolutely integrable means Lebesgue integrable.

0.3.1 Periodic functions

We find many periodic (or approximately periodic) functions in the real world, as for example a child on a swing (pendulum), planetary motions, or (atomic) clocks. In mathematics, we define a periodic function as follows.

Definition 0.6 (Periodic function) A function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a strictly positive constant $P > 0$ such that for all $x \in \mathbb{R}$

$$f(x+P) = f(x). \tag{0.1}$$

A constant $P > 0$ such that (0.1) is satisfied is called *a period*. The smallest such constant is called *the fundamental period*.

We often say 'the period' when we actually mean 'the fundamental period'. As we will see in Remark 0.7, if a period exists, then there are infinitely many periods. So actually it only makes sense to talk about 'a period' and not 'the period'.

Remark 0.7 1. We call the restriction $f_{|I} : I \to \mathbb{R}$ of a periodic function $f : \mathbb{R} \to \mathbb{R}$ of period P to any closed interval $I = [a, a+P]$, $a \in \mathbb{R}$, a *cycle*. We can obtain the entire function by 'copying and pasting' this cycle $f_{|I}$ to all intervals $[a+nP, a+(n+1)P]$, $n \in \mathbb{Z}$. Figure 0.3 illustrates this procedure: the solid part illustrates the graph of $f_{|I}$, whereas the dotted parts are the 'repetitions' on $[a-P, a]$ and $[a+P, a+2P]$. By comparing both graphs, assure yourself that this works independently of the choice of the particular interval.

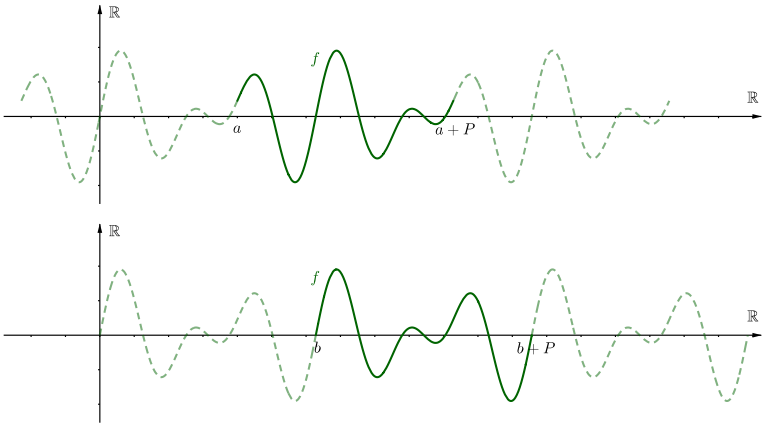


Figure 0.3: A periodic function $f : \mathbb{R} \to \mathbb{R}$ with period P can be constructed from any interval of length P.

2. From Figure 0.3 we can also intuit that any integer multiple of P is also a period of f. To be more precise, we can use induction over $\mathbb{N}$ to prove this claim. For this let f be a function of period P, so that the induction basis for $n = 1$, that is Equation (0.1), is satisfied. Let $n \in \mathbb{N}$ and assume $f(x+nP) = f(x)$ holds, this is the induction hypothesis (IH). Then

$$f(x+(n+1)P) = f(x+nP+P) \overset{(0.1)}{=} f(x+nP) \overset{\text{IH}}{=} f(x),$$

is satisfied. Similarly, we get

$$f(x-(n+1)P) \overset{(0.1)}{=} f(x-(n+1)P+P) = f(x-nP) \overset{\text{IH}}{=} f(x-nP+nP) = f(x).$$

By induction it follows that all elements in $\{nP \mid n \in \mathbb{Z}\}$ are periods of f. Hence, we have established the following result: if a period exists, it is *not unique* and there are infinitely many periods. Make sure you understand this basic example of a mathematical induction, it is very useful in many situations.

Exercise 0.8 *Explain how the 'copying and pasting' of a cycle mentioned in Remark 0.7 works. That is, given an $a \in \mathbb{R}$ and a function $g : I = [a, a+P] \to \mathbb{R}$ with $g(a) = g(a+P)$, construct a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period P such that $f_{|I} = g$. Or in other words, construct a function such that g is a cycle. Is g a* fundamental cycle*, that means, is P the fundamental period of f?*

Before we continue, we want to mention that periodicity can also be defined on subsets of $\mathbb{R}$. In practice for example, subsets could be intervals representing time, and thus are of the form $[0, T]$, $T > 0$. In this case, does it make sense to talk about periodicity if $2P \leq T$? Often, to simplify our work, it is convenient to 'cut' the interval to a length $T' = nP$ for some $n \in \mathbb{N}$. Also note that the second fact in Remark 0.7 is not true for functions only defined on intervals. But we could use our ability to create periodic functions on $\mathbb{R}$ from cycles on intervals.

Let us continue the discussion of periodic functions on $\mathbb{R}$. Probably the most familiar ones are sine and cosine.

Example 0.9 The functions sine and cosine are periodic with fundamental period 2π.

Exercise 0.10 *1. Is the tangent a periodic function?*

2. Show that the sum, the difference, and the product of two periodic functions $f, g : \mathbb{R} \to \mathbb{R}$ with the same fundamental period P are again functions of period P. Which further assumption do we need such that the same is true for quotients?
Hint: *Define the sum of the functions f and g pointwise by $z(x) = f(x) + g(x)$ for all $x \in \mathbb{R}$. Then use Equation* (0.1)*. Continue similarly for the rest.*

3. Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = sin(x) + sin(2\pi x)$ periodic? Can you find a condition such that the sum of two periodic functions on $\mathbb{R}$ is periodic?*

Next, we show an important property of periodic functions.

Lemma 0.11 *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period P. If f is integrable on some interval of length P, then it is integrable on all intervals of length P. In particular, the value of the integrals are equal, that is for any $a, b \in \mathbb{R}$ we have*

$$\int_a^{a+P} f(x)\,\mathrm{d}x = \int_b^{b+P} f(x)\,\mathrm{d}x, \tag{0.2}$$

Proof Without loss of generality (w.l.o.g.) we can assume $a = 0$ and $b \in [0, P]$. We split the integral on the interval $[b, b+P] = [b, P] \sqcup (P, b+P]$ and use periodicity to get

$$\begin{aligned}\int_b^{b+P} f(x)\,\mathrm{d}x &= \int_b^P f(x)\,\mathrm{d}x + \int_P^{b+P} f(x)\,\mathrm{d}x = \int_b^P f(x)\,\mathrm{d}x + \int_0^b f(x+P)\,\mathrm{d}x \\ &= \int_b^P f(x)\,\mathrm{d}x + \int_0^b f(x)\,\mathrm{d}x = \int_0^P f(x)\,\mathrm{d}x.\end{aligned}$$

□

Exercise 0.12 *Explain why it is enough to consider the case $a = 0$ and $b \in [0, P]$. Complete the proof of Lemma 0.11 for arbitrary $a, b \in \mathbb{R}$.*

Exercise 0.13 *Explain that if a function with period P is integrable on an interval of length P, then it is integrable on any interval of finite length.*

0.3.2 One-sided limits and discontinuities

In this section, we will introduce the notion of one-sided limits and discuss a few different types of discontinuities.

Definition 0.14 A one-sided limit is either of the two limits of a real-valued function $x \mapsto f(x)$ as x approach a specified value x_0 from above (the right) or from below (the left). The *left-sided limit* at a point x_0 is defined to be an $L^- \in \mathbb{R}$ such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ the following implication is satisfied: $0 < x_0 - x < \delta \implies |f(x) - L^-| < \epsilon$. As there is no single conventional notation, we list the most common ones. Provided the existence, we write for the left-sided limit

$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \nearrow x_0} f(x) = \lim_{x \uparrow x_0} f(x),$$

and for the right-sided limit we write

$$\lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \searrow x_0} f(x) = \lim_{x \downarrow x_0} f(x).$$

If both limits exist, but are not equal, then we say that the point x_0 is a point of discontinuity of the *first kind*, or simply, a point of *jump discontinuity*. If a function f has a jump discontinuity at x_0 , then the quantity

$$\delta = \lim_{x \downarrow x_0} f(x) - \lim_{x \uparrow x_0} f(x) \tag{0.3}$$

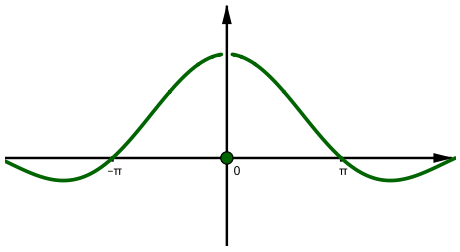
is called the *jump* of f at x_0.
If at least one of these limits does not exist, then the point x_0 is called a *point of discontinuity of the second kind*, or *essential discontinuity*.

Remark 0.15 1. If both one-sided limits exist and are equal, then we write the (two-sided) limit,

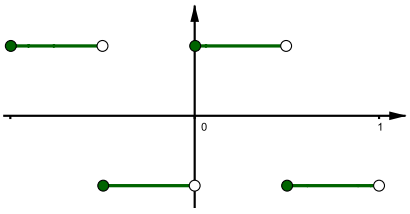
$$\lim_{x \uparrow x_0} f(x) = \lim_{x \downarrow x_0} f(x) = \lim_{x \to x_0} f(x).$$

2. We will see in Example 0.16 below that it often makes sense to allow the limits to take values $+\infty = \infty$ and $-\infty$. It is important to understand that $\{-\infty, +\infty\} \nsubseteq \mathbb{R}$ so that this is not entirely compatible with Definition 0.14. However, we can consider the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty] := \mathbb{R} \cup \{-\infty, +\infty\}$, and thus, allow positive and negative infinities.

Example 0.16 (Different kinds of discontinuities) Let us start with some 'nice' discontinuities exemplified in Figure 0.4.



(1) A removable discontinuity of $\frac{\sin(x)}{x}$



(2) Jump discontinuities of a square wave

Figure 0.4: Visualisation of removable and jump discontinuities

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

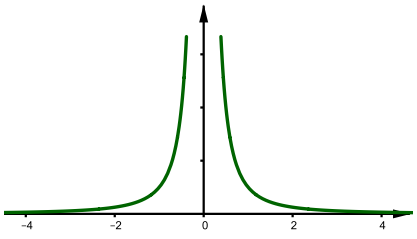
A priori we do not know what value $\frac{\sin(x)}{x}$ takes at 0, so we just set $f(0) = 0$. However, it turns out, using the series definition of the sine or L'Hôpital's rule, that $\lim_{x\uparrow 0} f(x) = \lim_{x\downarrow 0} f(x) = \lim_{x\to 0} f(x) = 1$. This kind of discontinuity is called a *removable discontinuity*, since by setting $f(0) = 1$ we could remove the discontinuity and make f continues at 0. (But we don't want to, we want an example of a removable discontinuity.)

2. Let us define a square wave as

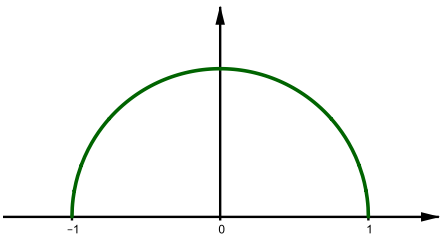
$$f(x) = \begin{cases} -1 & \text{if } \lfloor 2x \rfloor \pmod 2 = 0, \\ 1 & \text{else.} \end{cases}$$

For example, consider the jump discontinuity at $x_0 = 0$. We have $\lim_{x\uparrow x_0} f(x) = -1$ and $\lim_{x\downarrow x_0} f(x) = 1$. In particular, $\lim_{x\to x_0} f(x)$ does not exist, and the jump at $x_0 = 0$ is equal to $\delta = 2$. On the other hand, the jump at $x_0 = \frac{1}{2}$ is $\delta = -2$.

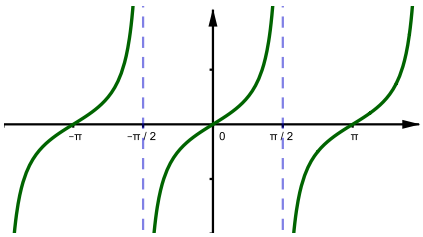
Let us look at some more examples.



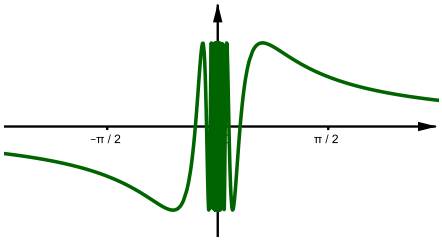
(3) An infinite discontinuity of $\frac{1}{x^2}$



(4) something



(5) Infinite discontinuities of $\tan(x)$



(6) An essential discontinuity of $\sin(\frac{1}{x})$

Figure 0.5: Visualisation of one-sided limits and discontinuities

3. Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = \frac{1}{x^2}$. Now, according to Definition 0.14, this function has neither one-sided limit. But if we allow infinite values we can write $\lim_{x\uparrow 0} f(x) = \lim_{x\downarrow 0} f(x) = +\infty$. Hence, even $\lim_{x\to 0} f(x) = +\infty$ is an acceptable notion. We call this kind of discontinuity an *infinite discontinuity*.

4. Consider $f : [-1, 1] \to [0, 1]$ defined by $f(x) = \sqrt{1 - x^2}$. We have $\lim_{x\downarrow -1} f(x) = 0$ and $\lim_{x\uparrow 1} f(x) = 0$, but $\lim_{x\uparrow -1} f(x)$ and $\lim_{x\downarrow 1} f(x)$ do not exist, since f is not defined outside $[-1, 1]$. So $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ do not exist.

5. Consider the tangent $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, defined by $\tan(x) = \frac{\sin(x)}{\cos(x)}$. We can define it on all intervals of the form $(\frac{2k-1}{2}\pi, \frac{2k+1}{2}\pi)$, for $k \in \mathbb{Z}$, and thus, we can define it on $\cup_{k\in\mathbb{Z}}(\frac{2k-1}{2}\pi, \frac{2k+1}{2}\pi)$. Now, neither of the one-sided limits exists at any of the discontinuity points. We find that as $x < x_0$ tends to x_0, the tangent tends to $+\infty$. Similarly, coming from the right, the tangent tends to $-\infty$. In this case, $\lim_{x\to x_0} \tan(x)$ is not defined.

6. Consider the function $f : \mathbb{R} \setminus \{0\} \to [0, 1]$ defined by $f(x) = \sin(\frac{1}{x})$. Now, neither $\lim_{x\uparrow 0} f(x)$ nor $\lim_{x\downarrow 0} f(x)$ exist, not even as infinity. We call this kind of discontinuity an *essential discontinuity* (as in 'this is essentially/substantially bad!').

Exercise 0.17 *Let us practice these new concepts in this exercise.*

1. Define the right-sided limit with help of the (ϵ, δ) notion, as we did in Definition 0.14 for the left-sided limit.

2. State the largest possible domains of the functions and calculate the left- and right-sided limits and state the jump sizes (if possible) of

(a) $f(x) = \lfloor x \rfloor$ *at all* $x_0 \in \mathbb{N}$,

(b) $f(x) = \frac{1-\cos(x)}{x^2}$ *at* $x_0 = 0$,

(c) $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ *at* $x_0 = 0$,

(d) $f(x) = \frac{x^2-3x}{x^2-9}$ *at* $x_0 = 3$ *and* $x_0 = -3$.

Hint: *L'Hôpital's rule can be handy in the second part of this exercise.*

Remark 0.18 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on the interval $[-\pi, \pi]$ with $f(-\pi) \neq f(\pi)$. When we periodically extend f from $[-\pi, \pi]$ to the whole axis, that means repeating $f|_{[-\pi,\pi)}$ on all intervals $[(k-1)\pi, (k+1)\pi)$, we produce jump discontinuities at the points $(2k+1)\pi$, $k \in \mathbb{Z}$, which have equal jumps sizes

$$\delta = f(-\pi) - f(\pi).$$

Note that this is not true for the tangent as we saw in Example 0.16. Why?

0.3.3 Even and odd functions

Let us briefly review functions which exhibit some sort of symmetry.

Definition 0.19 We say that $f : \mathbb{R} \to \mathbb{R}$ is *even* if for every $x \in \mathbb{R}$ we have

$$f(-x) = f(x).$$

We say f is *odd* if for every $x \in \mathbb{R}$ we have

$$f(-x) = -f(x).$$

This notion can also be used on subsets $A \subset \mathbb{R}$. We just have to make sure that for every $x \in A$ we also have $-x \in A$.

This definition implies that the graph of any even function is mirror-symmetric with respect to the y-axis. Similarly, graphs of odd functions are point-symmetric with respect to the origin $(0,0) \in \mathbb{R}^2$.

Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. As we can already guess from Figure 0.6, the integral of an even function f over an interval $[-\ell, \ell]$, $\ell \in \mathbb{R}_{\geq 0}$, can be calculated by only considering the interval $[0, \ell]$. Indeed, a short calculation yields that even functions satisfy for any $\ell \in \mathbb{R}_{\geq 0}$

$$\begin{aligned} \int_{-\ell}^{\ell} f(x)\,\mathrm{d}x &= \int_{-\ell}^{0} f(x)\,\mathrm{d}x + \int_{0}^{\ell} f(x)\,\mathrm{d}x = -\int_{0}^{-\ell} f(x)\,\mathrm{d}x + \int_{0}^{\ell} f(x)\,\mathrm{d}x \\ &= \int_{0}^{\ell} f(-x)\,\mathrm{d}x + \int_{0}^{\ell} f(x)\,\mathrm{d}x = 2\int_{0}^{\ell} f(x)\,\mathrm{d}x\,. \end{aligned} \tag{0.4}$$

Even 'better', if f is an odd function, a similar calculation yields that

$$\int_{-l}^{l} f(x)\,\mathrm{d}x = 0\,. \tag{0.5}$$

Example 0.20 The cosine is an even function, whereas the sine is an odd function. And indeed, for $\ell = \pi$ we have $\int_{-\pi}^{\pi} \cos(x)\mathrm{d}x = 2\int_{0}^{\pi} \cos(x)\mathrm{d}x = 0$ and $\int_{-\pi}^{\pi} \sin(x)\mathrm{d}x = 2 - 2 = 0$. Compare also Figure 0.6.

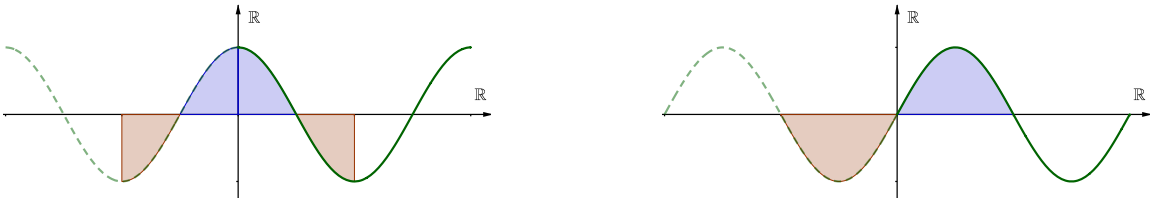


Figure 0.6: Integrals of cosine and sine, where blue indicates a part that with positive contribution whereas red indicates negative contribution to the integral.

The following two properties are direct consequences of the definition of even and odd functions.

Exercise 0.21 *Consider functions $\mathbb{R} \to \mathbb{R}$ and show the following two assertions.*

1. *The product of two even or two odd functions is an even function.*
2. *The product of an even and an odd function is an odd function.*

0.4 Linear Algebra

Linear algebra is concerned with vector spaces and linear mappings between these spaces. We will briefly revise vector spaces and their bases. We will continue with linear functions and their representation as matrices. We recall how the determinant, the characteristic polynomial, eigenvalues and eigenvectors can be computed. Finally we use these results to diagonalise matrices and calculate powers of them.

0.4.1 Vector spaces

Following the introduction of the most common sets, we can give our sets some structure. When we think of a set we often associate certain 'allowed actions' (or calculation rules) with this set. For instance, the well-known *multiplication and division first, then addition and subtraction*. It is

expected that you know all the basic 'calculation rules' for the sets introduced above, including exponentials and so on.

However, we briefly review Euclidean spaces. In particular, we are interested in $\mathbb{R}^n$ for $n \in \mathbb{N}$. Elements in $\mathbb{R}^n$ are called *vectors* and they are identified with n-tupels $x = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in \mathbb{R}$. $\mathbb{R}^1 = \mathbb{R}$ is the real line and $\mathbb{R}^2$ is the Euclidean plane as introduced above. You should be familiar with the following operations:

- Addition: For $x, y \in \mathbb{R}^n$ we have $x + y = (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{R}^n$.
- Scalar multiplication: For a *scalar* $a \in \mathbb{R}$ and a *vector* $x \in \mathbb{R}^n$ we have $ax = (ax_1, \ldots, ax_n) \in \mathbb{R}^n$.

For an arbitrary set to be a vector space the set and the operations of summation and scalar multiplication have to satisfy the following axioms:

- For addition: associativity, commutativity, existence of an identity element, existence of inverse elements.
- For scalar multiplication: Compatibility of multiplications, existence of an identity element, distributivity of vector and scalar addition.

Exercise 0.22 *Show that $\mathbb{R}^n$ together with the addition and scalar multiplication defined above is indeed a vector space.*
Hint: *If you cannot recall what the axioms mean, Wikipedia is a good place to start searching.*

Let us define one more useful operation:

- Inner product[8]: For $x, y \in \mathbb{R}^n$ we have $x \cdot y = \sum_{j=1}^{n} x_j y_j \in \mathbb{R}$.

The magnitude (or length) of a vector x is denoted by $\|x\|$ and is defined as $\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{j=1}^{n} x_j^2}$. For Euclidean vectors, which we can think of as actual arrows (at least for $n \leq 3$), the inner product can also be written as

$$x \cdot y = \|x\| \, \|y\| \cos(\vartheta),$$

where ϑ is the angle between x and y. This is particularly useful if we want to determine the angle between two vectors, which is given by the inverse

$$\vartheta = \arccos\left(\frac{x \cdot y}{\|x\| \, \|y\|}\right).$$

0.4.2 Bases of vector spaces

Vector spaces have a basis with which each element can be represented. For example, $\mathbb{R}^n$ comes with what is known as the *standard basis*, which is a collection of unit vectors $\{e_1, \ldots, e_n\} \subset \mathbb{R}^n$, defined as

$$e_1 = (1, 0, 0 \ldots, 0, 0)^\intercal, \; e_2 = (0, 1, 0, \ldots, 0, 0)^\intercal, \ldots, \; e_n = (0, 0, 0, \ldots, 0, 1)^\intercal.$$

The operator '$^\intercal$' is called the *transpose* of a matrix and it flips the row and column indices of a matrix. We usually think of vectors as *column vectors* and we use the transpose here to save space. As an example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad A^\intercal = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

[8] Also called *dot product* or *scalar product*.

The unit vectors are normalised, $\|e_1\| = \ldots = \|e_n\| = 1$, and they are pairwise orthogonal, that means for $1 \leq k, \ell \leq n$, $k \neq \ell$ we have $e_k \cdot e_\ell = 0$. They represent the axes of a *Cartesian coordinate system* and each element in $\mathbb{R}^n$ can be written as a linear combination of them. For example in $\mathbb{R}^2$, each point on the unit circle can be written as $x = (\cos(\vartheta), \sin(\vartheta))^\intercal$. So splitting it up in horizontal and vertical axis, we can write $x = \cos(\vartheta)e_1 + \sin(\vartheta)e_2$.

Now, for a *given basis* (usually the standard basis in our case) a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ can be *uniquely* represented by an $m \times n$-matrix $A \in \mathbb{R}^{m \times n}$. We denote the entries of a matrix with two indices as $A = (A_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, where the first one denotes the row and the second the column number. Note that a different basis will in general give a different matrix.
Consider for example the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $x \mapsto (x_1, -x_2)^\intercal$, the reflection across the x-axis. We can represent this linear function by the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, since $Ax = (x_1, -x_2)^\intercal$.

Exercise 0.23 *Given the standard basis of $\mathbb{R}^n$, find matrices which represent the functions*

- $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ *with* $x \mapsto ax$, *for some* $a \in \mathbb{R}$,
- $f_2 : \mathbb{R}^3 \to \mathbb{R}^2$ *with* $x \mapsto (x_1 + 2x_2, x_2 - 3x_3)^\intercal$.

Is it possible to find a matrix which represents $f : \mathbb{R}^2 \to \mathbb{R}$ with $x \mapsto \|x\| = \sqrt{x_1^2 + x_2^2}$?
Hint: *Why not?*

Exercise 0.24 *Recall some matrix operations. Let $A, B \in \mathbb{R}^{n \times n}$ and let $a \in \mathbb{R}$.*

- *Calculate aA, $A + B$, AB.*
- *Show that $(A^\intercal)^\intercal = A$, $(A + B)^\intercal = A^\intercal + B^\intercal$, and $(AB)^\intercal = B^\intercal A^\intercal$.*
- *Show that in general $AB \neq BA$.*
- *Write the dot product $x \cdot y$ of two vectors $x, y \in \mathbb{R}^n$ with help of the transpose operator.*

Matrices are very useful to study linear functions. Moreover, they can be used to study *systems of linear equations* (or *linear systems*). A system of m linear equations with n unknown variables is defined as

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \,. \end{aligned}$$

This system can be written more compactly in matrix notation. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be the $m \times n$-matrix defined by the coefficients of the system, define $b = (b_1, \ldots, b_m)^\intercal$ by the constant terms and let $x = (x_1, \ldots, x_n)^\intercal$. Then the linear system from above takes the form

$$Ax = b \,.$$

This equation potentially has the solution $x = A^{-1}b$, if A is *invertible*, this means, if the inverse A^{-1} of A exists. The inverse of an $n \times n$-matrix A is denoted by A^{-1} and it is also an $n \times n$-matrix defined as the left and right inverse $AA^{-1} = A^{-1}A = \mathbb{1}_n \in \mathbb{R}^{n \times n}$, where $\mathbb{1}_n$ (or I_n) is the $n \times n$ *identity matrix*

$$\mathbb{1}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

0.4.3 Determinant

The *determinant* of a real-valued square matrix A is a value $\det A \in \mathbb{R}$ (or $\det(A), |A|$) which in a loose sense represents the scaling factor of A. Formally, it is defined as the value of a multi-linear (linear in each column), alternating (equal 0 if two columns are equal), and normalised ($\det \mathbb{1}_n = 1$) map from the space of all square matrices to the underlying field (in our case $\mathbb{R}$).
We will mainly use it to determine whether a system of linear equation is solvable.

Proposition 0.25 *A square matrix is invertible if and only if its* determinant *is not equal* 0*. In this case, we have* $\det A^{-1} = (\det A)^{-1}$.

Let us recall how determinants can be calculated for (2×2), (3×3), and even larger matrices.

Proposition 0.26 *Let* $A = (a_{ij})_{1\le i,j\le n}$ *be an* $n\times n$*-matrix.*

1. *For* $n = 2$ *the determinant of* A *is given by*

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}\,.$$

2. *For* $n = 3$ *we have*

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\,.$$

3. *For arbitrary* n *we can use the* Leibniz formula*. Actually, we will use the* Laplace formula (Laplacescher Entwicklungssatz)*, which essentially is a procedure to evaluate the Leibniz formula. For a fixed column* $1 \le j \le n$ *it takes the form*

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}\,,$$

whereas for a fixed row $1 \le i \le n$ *we get*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}\,.$$

Here, A_{ij} *is the* $(n-1)\times(n-1)$*-matrix that results from* A *by removing the* i*-th row and the* j*-th column. The term* $\det A_{ij}$ *is called a* minor *of* A*. The choice of the row or the column we want to use does not matter, so it often makes sense to choose it such that many of the coefficients* a_{ij} *are zeros.*

Example 0.27 Let us look at an example. Calculate the determinant of the following matrix.

$$A = \begin{pmatrix} 1 & 0 & -3 & a \\ 3 & -4 & 4 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 5 & -2 \end{pmatrix}.$$

We notice that this matrix has three zeros in the third row. Therefore, to make our lifes easy, we choose the third row as the fixed row. The Laplace formula takes the form

$$\det A = \sum_{j=1}^{4} (-1)^{3+j} a_{3,j} \det A_{3,j}\,.$$

Since $a_{3,1} = a_{3,2} = a_{3,3} = 0$, we only have to consider the (3×3)-matrix $A_{3,4}$, which we get by removing the third row and the forth column of A,

$$A_{3,4} = \begin{pmatrix} 1 & 0 & -3 & a \\ 3 & -4 & 4 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 5 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 3 & -4 & 4 \\ 3 & 1 & 5 \end{pmatrix}.$$

We can now continue with Laplace to calculate the minor $\det A_{3,4}$, or we simply use the formula for the determinant of a (3×3)-matrix. We choose the latter approach and get

$$\det A = (-1)^{3+4} \cdot 1 \cdot \begin{vmatrix} 1 & 0 & -3 \\ 3 & -4 & 4 \\ 3 & 1 & 5 \end{vmatrix} = -(-20 + 0 - 9 - 36 - 4 - 0) = 69\,.$$

Make sure you feel familiar with determinants and how they can be calculated. Here are a few short exercises to practice on.

Exercise 0.28 *Calculate the determinants of the following matrices and state for which $a \in \mathbb{R}$ each one is invertible.*

1. $$A = \begin{pmatrix} 3 & a \\ 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} a & 2 \\ 3a & 6 \end{pmatrix} \qquad C = \begin{pmatrix} a & 2 \\ \frac{1}{2} & 4a \end{pmatrix}$$

2. $$D = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & a \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 9 & 9 \\ 0 & 1 & 9 \\ 0 & 0 & a \end{pmatrix}$$

3. $$F = \begin{pmatrix} 2 & 4 & 1 & -3 \\ a & 3a & 0 & 0 \\ 3 & 5 & 2 & -1 \\ 1 & 0 & -a & 7 \end{pmatrix}$$

Determinants have many nice properties. We summarise some important ones and we discuss yet another way to calculate the determinant of the matrix A in Example 0.27.

Proposition 0.29 *Let $A, B \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}$, then we have*

1. *The determinant is a multiplicative map, this means* $\det(AB) = \det(A)\det(B)$.
2. *If A is a triangular matrix, this means if $a_{ij} = 0$ for all $i < j$ (or $i > j$), then* $\det A = \prod_{i=1}^{n} a_{ii}$.
3. *If B results from A by exchanging two rows (or columns), then* $\det B = -\det A$.
4. *If B results from A by adding a multiple of a row (or column) to* another *row (or column), then* $\det B = \det A$.
5. *If B results from A by multiplying a row (or column) by the constant a, then* $\det B = a \det A$.

Now Proposition 0.29 Assertion 2 gives us an elegant way to calculate determinants. However, to apply this result, we must have a triangular matrix. Luckily, we can often apply what is known as the *Gaussian elimination method (Gaußsches Eliminationverfahren)*.

Exercise 0.30 *Explain the steps used to calculate the determinant of the matrix A in Example 0.27. In particular, explain how assertions 2, 3, and 4 in Proposition 0.29 are applied.*

$$\begin{vmatrix} 1 & 0 & -3 & a \\ 3 & -4 & 4 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 5 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -3 & a \\ 3 & -4 & 4 & 2 \\ 3 & 1 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -3 & a \\ 3 & -4 & 4 & 2 \\ 0 & 5 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -3 & a \\ 0 & -4 & 13 & 2-3a \\ 0 & 5 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & -3 & a \\ 0 & -4 & 13 & 2-3a \\ 0 & 0 & \frac{65}{4}+1 & -\frac{6+15a}{4} \\ 0 & 0 & 0 & 1 \end{vmatrix} = -\left(1 \cdot (-4) \cdot \left(\frac{65}{4}+1\right) \cdot 1\right) = 69\,.$$

Exercise 0.31 *Use assertions 1 and 2 in Proposition 0.29 to show that* $\det(aA) = a^n \det A$*, for* $a \in \mathbb{R}$.

0.4.4 Eigenvectors, eigenvalues, and the characteristic polynomial

Consider a linear transformation from a vector space to itself. In our case, we consider a square matrix, that is a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$. If we find a non-zero vector $x \in \mathbb{R}^n$ such that the vector Ax is a scalar multiple of it, that is if,

$$Ax = \lambda x\,, \tag{0.6}$$

then x is an eigenvector of this matrix and λ is the eigenvalue corresponding to this eigenvector.

Exercise 0.32 *Consider the following* eigenvalue equation *and find the eigenvalue* λ *corresponding to the given eigenvector.*

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \lambda \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The simplest way to find eigenvalues of a matrix (if you do not know the eigenvectors) is with help of the characteristic polynomial. The eigenvalue equation (0.6) can be written as $\lambda x - Ax = (\lambda \mathbb{1}_n - A)x = 0$. Since x is non-zero, the matrix $\lambda \mathbb{1}_n - A$ must be *singular* (not invertible) and thus, its determinant must be 0. The *characteristic polynomial* of a square matrix $A \in \mathbb{R}^{n \times n}$ is then defined as the determinant of this matrix

$$p_A(\lambda) = \det(\lambda \mathbb{1}_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\,, \tag{0.7}$$

and hence, the roots of this polynomial are therefore the eigenvalues of the matrix.

Example 0.33 Consider the matrix $A = \begin{pmatrix} 3 & a \\ 4 & 0 \end{pmatrix}$ from Exercise 0.28. The characteristic polynomial is given by

$$\det(\lambda \mathbb{1}_n - A) = \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 3 & a \\ 4 & 0 \end{pmatrix}\right) = \begin{vmatrix} \lambda - 3 & -a \\ -4 & \lambda \end{vmatrix} = \lambda(\lambda - 3) - 4a = \lambda^2 - 3\lambda - 4a\,.$$

The roots of this polynomial are $\lambda_1 = \frac{3 - \sqrt{16a+9}}{2}$ and $\lambda_2 = \frac{3 + \sqrt{16a+9}}{2}$. For $a = 1$, we have $\lambda_1 = -1$ and $\lambda_2 = 4$. Solving the eigenvalue equation with these eigenvalues we get the corresponding eigenvectors as multiples of

$$x_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Beside this useful application, it has some interesting properties. For example, the coefficient of λ^n is always 1, the coefficient a_{n-1} of λ^{n-1} is $-\mathrm{tr}(A) = -\sum_{j=1}^{n} a_{jj}$, and the coefficient a_1 is given by $(-1)^n \det A$.

0.4.5 Diagonalisable matrices

A square matrix A over $\mathbb{R}$ (or $\mathbb{C}$, or any field) is called a *diagonal matrix* if all entries outside the leading diagonal are zero. We often write an $(n \times n)$-diagonal matrix A as

$$A = \mathrm{diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{pmatrix}.$$

A square matrix A is called *diagonalisable* if it is *similar* to a diagonal matrix, that means if there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Proposition 0.34 *A matrix $A \in \mathbb{R}^{n \times n}$ is diagonalisable if and only if the eigenvectors of A form a basis of $\mathbb{R}^n$. In this case, the columns of P can be defined as the eigenvectors of A.*

Example 0.35 Consider the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$ from Example 0.33. We have already established that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$ with corresponding eigenvectors $x_1 = (1, -4)^\intercal$ and $x_2 = (1, 1)^\intercal$. Next, define the columns of P with these eigenvectors

$$P = (x_1, x_2) = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}.$$

We immediately see that $\det P = 5 \neq 0$ and thus, the eigenvectors are linearly independent and P is invertible. Using for example Gauß-elimination we quickly find its inverse

$$P^{-1} = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}.$$

The result immediately follows

$$P^{-1}AP = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Note that the order of the columns in P does not matter and it will only affect the order of the eigenvalues in the resulting diagonal matrix.

Diagonal matrices are usually easy to handle. For example, powers of a diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$ are simply given by

$$A^k = \mathrm{diag}(a_1^k, \ldots, a_n^k).$$

From here we get an elegant way to compute the powers of a diagonalisable matrix A. Let $D = P^{-1}AP$ denote the resulting diagonal matrix, then $A = PDP^{-1}$. Hence,

$$A^k = (PDP^{-1})^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD(P^{-1}P)D(P^{-1}P) \cdots DP^{-1} = PD^kP^{-1}.$$

Since D is diagonal, its powers are given as above. This procedure is usually much faster than calculating A^k directly.

Example 0.36 Just imagine how labourious it would be if someone asked you to calculate some specific very large powers of A from Example 0.35. You would start

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 13 & 3 \\ 12 & 4 \end{pmatrix}, \quad A^3 = A^2 \cdot A = \begin{pmatrix} 51 & 13 \\ 52 & 12 \end{pmatrix}, \quad A^4 = A^3 \cdot A = \begin{pmatrix} 205 & 51 \\ 204 & 52 \end{pmatrix}, \dots$$

and soon get tired. But you could actually use the clever and much more rewarding way. From Example 0.35 we know that A is diagonalisable and the diagonal matrix is given by $D = \mathrm{diag}(-1, 4)$. The k-th power is quickly calculated as $D^k = \mathrm{diag}((-1)^k, 4^k)$. Hence,

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} (-1)^k & 0 \\ 0 & 4^k \end{pmatrix} \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4^{k+1} + (-1)^k & 4^k + (-1)^{k+1} \\ 4^{k+1} + 4 \cdot (-1)^{k+1} & 4^k + 4 \cdot (-1)^k \end{pmatrix} \\ &= \frac{1}{5} \left(4^k \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} + (-1)^k \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \right). \end{aligned}$$

You are now ready to plug your specific k in and be on your way to do some more maths.

0.4.6 Matrix exponential

Let $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$). The exponential of A is denoted by e^A or $\exp(A)$ and is (in analogy to the real case) defined via the power series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \in \mathbb{R}^{n\times n} \quad (\text{ or } \mathbb{C}^{n\times n}).$$

The nice thing is that this series converges for all real (and complex) matrices and thus, the matrix exponential is always well-defined. Let us briefly outline why this is true. Since $\mathbb{R}^{n\times n}$ (and $\mathbb{C}^{n\times n}$) are complete it is sufficient to show convergence in norm, and since $\mathbb{R}^{n\times n}$ (and $\mathbb{C}^{n\times n}$) are finite-dimensional vector spaces, all norms define the same topology. So taking a sub-multiplicative norm, we establish $0 \leq \|\frac{A^k}{k!}\| \leq \frac{\|A\|^k}{k!}$, for all $k \in \mathbb{N}_0$. Thus,

$$\left\| \sum_{k=0}^{\infty} \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!}.$$

But $\|A\| \in \mathbb{R}$ and we know that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all $x \in \mathbb{R}$, hence, e^A is well-defined. With this we have also established that $\|e^A\| \leq e^{\|A\|}$.

In fact, there are a number of properties of the usual exponential on $\mathbb{R}$ (or $\mathbb{C}$) which also hold for the matrix exponential.

Proposition 0.37 *Let $A, B \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) and let $a, b \in \mathbb{R}$ (or $\mathbb{C}$).*

1. *Let $0 \in \mathbb{R}^{n\times n}$ be the $(n \times n)$-zero matrix, then $e^0 = \mathbb{1}_n$.*
2. *We have $e^{aA}e^{bA} = e^{(a+b)A}$.*
3. *If A, B commute, that means $AB = BA$, then $e^{A+B} = e^A e^B$.*
4. *The exponential of A is always invertible and the inverse is $(e^A)^{-1} = e^{-A}$.*

Proof We will only prove assertions 1, 2, and 4. Assertion 3 can be shown more elegantly after we discuss the exponential map some more.

1. Since for any matrix $A \in \mathbb{R}^{n\times n}$ we have $A^0 = \mathbb{1}_n$, the assertion follows directly.

2. We have

$$e^{aA}e^{bA} = \sum_{k=0}^{\infty}\frac{(aA)^k}{k!}\sum_{\ell=0}^{\infty}\frac{(bA)^\ell}{\ell!} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{a^k b^\ell A^{k+\ell}}{k!\ell!} \overset{\substack{n=k+\ell\\ \ell=n-k}}{=} \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{a^k b^{n-k}A^n}{k!(n-k)!}\cdot\frac{n!}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{A^n}{n!}\frac{n!}{k!(n-k)!}a^k b^{n-k} = \sum_{n=0}^{\infty}\frac{A^n}{n!}(a+b)^n = e^{(a+b)A}.$$

4. From assertions 1 and 2 we immediately get

$$e^{-A}e^{A} = e^{A}e^{-A} = e^{(1-1)A} = e^0 = \mathbb{1}_n.$$

hence, e^{-A} must be the inverse of e^A. □

With Assertion 4 of Proposition 0.37 we established that $\exp : \mathbb{C}^{n\times n} \to \mathrm{GL}_n(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ is called the *general linear group over* $\mathbb{C}$ *of degree* n and is the set of all invertible matrices together with the usual matrix multiplication, $\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid \det A \neq 0\}$.

For some $A \in \mathbb{R}^{n\times n}$ consider the function

$$f : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C}) \text{ with } t \mapsto e^{tA}.$$

It turns out that this function is actually a smooth curve in $\mathrm{GL}_n(\mathbb{C})$ and its (component-wise) derivative at a point $t \in \mathbb{R}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tA} = Ae^{tA} = e^{tA}A.$$

We already established the well-definedness of the matrix exponential. This also means that its component-wise series converge. Hence, we can exchange the order of summation and differentiation and we get (component-wise)

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tA} = \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^k A^k}{k!}\right) = \sum_{k=1}^{\infty}\frac{\cancel{k}t^{k-1}A^k}{\cancel{k}(k-1)!} = A\sum_{k=1}^{\infty}\frac{t^{k-1}A^{k-1}}{(k-1)!} = Ae^{tA}. \tag{0.8}$$

This result will be very useful when we solve systems of differential equations. See Theorem 2.5.

1 Fourier Series

The Fourier series, named in honour of Jean-Baptiste Joseph Fourier, is a series of sines and cosines (or equivalently complex exponentials) which can be used to represent periodic functions. It turns out that a lot of periodic functions can be represented as a series of sines and cosines and practical applications reach from signal processing, as for example noise minimisation and compression of music, for example mp3, to solving partial differential equations.

1.1 How the Fourier series naturally arises

The Fourier series was initially suggested in order to solve certain differential equations. We will see how the well-known *heat equation* motivates the theory of Fourier series.
Let us agree that temperature is a function of time and space. For example, turning on the engine in your car will cause the metal to heat up over time. But the temperature is not the same on all the parts, so it also depends on the particular location we look at. The heat equation is a partial differential equation and it describes how the temperature changes over time.

We study the one dimensional heat equation as a simplification of the heat propagation in a thin metal rod of length $\ell > 0$. Consider Figure 1.1 and let $u : [0,\ell] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ denote the temperature of the rod.

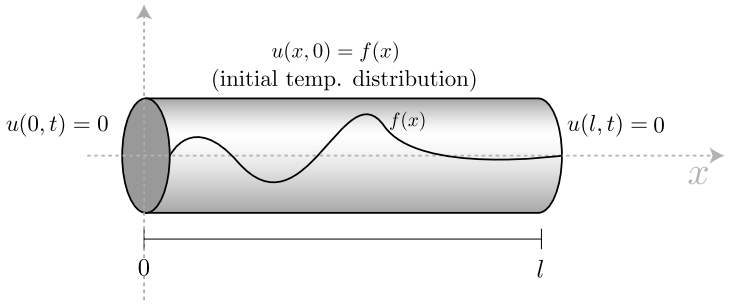


Figure 1.1: Idealized physical setting for heat conduction in a rod with homogeneous boundary conditions. Source: `https://upload.wikimedia.org/wikipedia/commons/9/97/Temp_Rod_homobc.svg`

Assume that u is at least twice differentiable in x and one time differentiable in t. The heat equation in one dimension with *Dirichlet boundary conditions* $u(0,t) = u(\ell,t) = 0$ and initial temperature distribution $u(\cdot,0) = f : [0,\ell] \to \mathbb{R}$ is given by

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{or shorter } \partial_t u = \alpha \partial_{xx} u\,.$$

If $f \equiv 0$, then $u \equiv 0$ is the unique solution. But we are not interested in the trivial solution. Let us take the guess that u is of the form $u(x,t) = X(x)T(t)$. Then the heat equation takes the form

$$X(x)T'(t) = \alpha X''(x)T(t)\,.$$

To simplify, we can assume without loss of generality (w.l.o.g.) that $\alpha = 1$ (redefine $\bar{X}(x) = X(\frac{1}{\sqrt{\alpha}}x)$, since $\alpha \neq 0$). This is equivalent to (assuming we do not divide by zero)

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}\,. \tag{1.1}$$

Now the left hand side of Equation (1.1) is a function of x, whereas the right hand side only depends on t. This is only possible if this expression is constant, that means if there is a constant $\lambda \in \mathbb{R}$ such that $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda$, with the conventional minus sign. This gives us two ordinary differential equations

$$X''(x) + \lambda X(x) = 0\,,$$
$$T'(t) + \lambda T(t) = 0\,.$$

We quickly find that $T(t) = T_0 e^{-\lambda t}$, for some constant T_0. If $T_0 = 0$, then we have the trivial solution again. So let us assume $T_0 \neq 0$. In this case, the Dirichlet boundary conditions dictate that $X(0) = X(\ell) = 0$. This is an example of a Sturm-Liouville problem. Here, we differentiate three cases, $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. It turns out that only the case $\lambda > 0$ gives us non-trivial solutions of the form

$$X(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)\,.$$

The boundary conditions give us $C_2 = 0$ and $C_1 \sin(\sqrt{\lambda}\ell) = 0$. We do not want $C_1 = 0$ (trivial solution), but we can choose $\lambda = \frac{n^2\pi^2}{\ell^2}$, for $n \in \mathbb{Z}$. Hence, for all $n \in \mathbb{N}$ and constants C_n the functions $X_n(x) = C_n \sin(\frac{n\pi x}{\ell})$ are solutions. Similarly we now get $T_n(t) = D_n \exp\left(-\frac{n^2\pi^2 t}{\ell^2}\right)$, for some constants D_n. Combining T_n and X_n and setting $b_n = C_n D_n$ we get for $n \in \mathbb{N}$ solutions of the heat equation as

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{\ell}\right)\exp\left(-\frac{n^2\pi^2 t}{\ell^2}\right),$$

which satisfy $u_n(0,t) = u_n(\ell,t) = 0$. We are almost done. The only thing we have not yet considered is the initial temperature distribution $u(\cdot,0) = f$. We notice that the exponential equals 1 for $t = 0$, so that $u_n(x,0) = b_n \sin\left(\frac{n\pi x}{\ell}\right)$. Since (finite) linear combinations of solutions are again solutions, we could try and combine all the solutions to form a series (one has to show that this also works)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)\exp\left(-\frac{n^2\pi^2 t}{\ell^2}\right),$$

so that the initial temperature distribution must be given by the following series,

$$f(x) \stackrel{!}{=} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

It turns out that we can indeed sometimes find coefficients b_n such that the series converges and is equal to f. We call such a series the *Fourier series of* f. The problem of finding the coefficients and studying convergence properties of the series is part of the Fourier analysis.

1.2 The building blocks: simple harmonic motion

In applications, we often encounter periodic functions $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = A \sin(\omega x + \varphi), \tag{1.2}$$

where A, ω and ϕ are constants in $\mathbb{R}$. This function is called a *harmonic of amplitude* $|A|$*, (angular) frequency* ω*, and the initial phase* ϕ. The period of such a harmonic is $P = \frac{2\pi}{\omega}$, since for any $x \in \mathbb{R}$ we have

$$A \sin\left(\omega\left(x + \frac{2\pi}{\omega}\right) + \phi\right) = A \sin\left((\omega x + \phi) + 2\pi\right) = A \sin(\omega x + \phi)\,. \tag{1.3}$$

The terms *amplitude*, *frequency*, and *initial phase* originate from mechanical problems involving simple oscillatory motions, that is *simple harmonic motions*, as for example, a frictionless spring-mass system. The term harmonic is strictly speaking any term in the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. The series gets its name from harmonics in music, since the overtones of a string are $\frac{1}{n}$ of the fundamental wavelength.

Example 1.1 Consider an E string on a violin or a guitar, which has some fundamental frequency. Dividing the string into two equal parts, we double the fundamental frequency and get the second harmonic, which sounds an octave higher. The third harmonic with three times the fundamental frequency gives a perfect fifth above the second harmonic, which in this case would be a B. Can you continue with this list?

Adding single sinusoids from Equation (1.2) with different A, ω, φ together we can build more involved periodic functions. It turns out that taking the sum over just the sine leads to complicated calculations and that it is easier to add the cosine into the picture. Let $N \in \mathbb{N}$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big(a_n \cos(nx) + b_n \sin(nx) \Big), \tag{1.4}$$

where $a_0, a_n, b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. The constant term $\frac{a_0}{2}$ is referred to as the direct current (DC) component in electrical engineering, and also here, the fraction $\frac{1}{2}$ will simplify notation when we consider the complex representation. Note that the harmonics in Equation (1.4) have periods $\frac{2\pi}{n}$ for $n \in \{1, \dots, N\}$, respectively.

Exercise 1.2 *Show that the sum in Equation* (1.4) *has fundamental period* 2π.

So a sum as in Equation (1.4) defines a function of period 2π. The aim of this chapter is to investigate the other direction: given a function $f : \mathbb{R} \to \mathbb{R}$ of period 2π, can we find a sum or series which represents f? We will see that this is sometimes possible, but not always. The length of the period does not matter for the representation, but for simplicity we will start with periods of length 2π.

1.3 Building with the building blocks

Let us start with the assumption that f indeed yields such a representation and investigate what rules this representation must obey. For one, we could try and find the coefficients a_0, a_n, b_n in terms of our function f. The following lemma is useful for the derivation of these coefficients.

Lemma 1.3 *Let* $n, m \in \mathbb{N}$. *The following auxiliary integrals are used in the derivation of the Fourier coefficients.*

1. $\displaystyle\int_0^{2\pi} \cos(nx)\,\mathrm{d}x = 0$
2. $\displaystyle\int_0^{2\pi} \sin(nx)\,\mathrm{d}x = 0$
3. $\displaystyle\int_0^{2\pi} \sin(nx)\cos(mx)\,\mathrm{d}x = 0$
4. $\displaystyle\int_0^{2\pi} \cos(nx)\cos(mx)\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$
5. $\displaystyle\int_0^{2\pi} \sin(nx)\sin(mx)\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$

Exercise 1.4 *Make sure you feel confident calculating these integrals and that you know all necessary steps. What changes if we consider* $\sin(Pnx)$ *and* $\cos(Pnx)$ *for some* $P > 0$*? Compare your thoughts to Exercise 1.17.*

In the first step of the derivation of the coefficients, we integrate both sides of Equation (1.4) over the interval $[0, 2\pi]$. Since the integral is linear, we can exchange the sum and the integral and then use Lemma 1.3 to get

$$\begin{aligned}\int_0^{2\pi} f(x)\,\mathrm{d}x &= \int_0^{2\pi} \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)\,\mathrm{d}x \\ &= a_0\pi + \sum_{n=1}^{N} a_n \underbrace{\int_0^{2\pi} \cos(nx)\,\mathrm{d}x}_{=0,\forall n\in\{1,\dots,N\}} + b_n \underbrace{\int_0^{2\pi} \sin(nx)\,\mathrm{d}x}_{=0,\forall n\in\{1,\dots,N\}} = a_0\pi\,.\end{aligned}$$

Consequently, a_0 is given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x. \tag{1.5}$$

In the second step, we multiply both sides of (1.4) by $\cos(mx)$, for some fixed $m \in \{1,\dots,N\}$, integrate the resulting expression from 0 to 2π, and again use Lemma 1.3. We obtain

$$\begin{aligned}\int_0^{2\pi} f(x)\cos(mx)\,\mathrm{d}x &= \frac{a_0}{2}\underbrace{\int_0^{2\pi} \cos(mx)\,\mathrm{d}x}_{=0,\forall m\in\{1,\dots,N\}} \\ &\quad + \sum_{n=1}^{N} a_n \underbrace{\int_0^{2\pi} \cos(nx)\cos(mx)\,\mathrm{d}x}_{=\pi \text{ if } m=n \text{ and } =0 \text{ else}} + b_n \underbrace{\int_0^{2\pi} \sin(nx)\cos(mx)\,\mathrm{d}x}_{=0,\forall m,n\in\{1,\dots,N\}} \\ &= a_m\pi\,.\end{aligned}$$

Therefore,

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\,\mathrm{d}x, \quad \text{for } m \in \{1,\dots,N\}. \tag{1.6}$$

Similarly, multiplying both parts of Equation (1.4) with $\sin(mx)$ and integrating from 0 to 2π we get

$$\begin{aligned}\int_0^{2\pi} f(x)\sin(mx)\,\mathrm{d}x &= \frac{a_0}{2}\int_0^{2\pi} \sin(mx)\,\mathrm{d}x \\ &\quad + \sum_{n=1}^{N} a_n \int_0^{2\pi} \cos(nx)\sin(mx)\,\mathrm{d}x + b_n \int_0^{2\pi} \sin(nx)\sin(mx)\,\mathrm{d}x \\ &= b_m\pi\,.\end{aligned}$$

It follows that

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)\,\mathrm{d}x, \quad \text{for } m \in \{1,\dots,N\}. \tag{1.7}$$

We can conclude that if a function f of period 2π can be written as a trigonometric sum (1.4), then the coefficients must be given by (1.5), (1.6), and (1.7). However, we have not yet answered the question whether all periodic function can be written as a sum. This question will be addressed in the following section.

1.4 Fourier Series

For the derivation of the coefficients above we used the assumption that a function f is represented by a sum of the form (1.4). Since it was just a finite series (a sum), we were allowed to exchange the integral and the sum. In general, this procedure is not applicable to series and f must satisfy certain conditions. It is common to write

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big)$$

to indicate some connection between the function f on the left-hand side and the series on the right-hand side. In our case, we indicate with $\sim$ that the coefficients a_0, a_n, and b_n are defined by the Fourier-coefficients in (1.5), (1.6), and (1.7), respectively. In this sense, the Fourier series is always defined as long as the integrals exist. However, this does not mean that f and the series are a priori equal, so we refrain from using the equal sign.

To study this in more detail, one has to consider the *partial sum*

$$S_{f,N}(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \Big(a_n \cos(nx) + b_n \sin(nx) \Big).$$

We want to find out under which conditions and in which sense $S_{f,N}$ converges to f. For real-world applications we are usually interested in pointwise, uniform, and L^p convergence.

The following results, names after Peter Gustav Lejeune Dirichlet, states sufficient conditions on f for pointwise convergence of its Fourier series.

Theorem 1.5 (Dirichlet conditions) *If $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function which is is absolutely integrable over one period, has a finite number of extrema and a finite number of non-infinite discontinuities in any given interval, then for all $x \in \mathbb{R}$ the Fourier series of f converges (pointwise) to the arithmetic mean of the left and right limit of f at x,*

$$\lim_{N\to\infty} S_{f,N}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) = \frac{1}{2} \left(\lim_{y \downarrow x} f(y) + \lim_{y \uparrow x} f(y) \right).$$

So any 2π-periodic function which satisfies these conditions can be represented by its Fourier series. Notice however, that at points of discontinuity, the value of the Fourier series is the midpoint of the values at the discontinuity. Compare this result to Figure 1.3.

Corollary 1.6 *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the Dirichlet conditions in Theorem 1.5 and assume that f is continuous on $\mathbb{R}$, then the Fourier series of f converges and is equal to f.*

Remark 1.7 The Dirichlet conditions are satisfied if f has bounded variation over one period, that is, if its *total variation* is finite,

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=1}^{n_P} |f(x_{i+1} - f(x_i)| < \infty,$$

where the supremum is taken over the set $\mathcal{P}$ of all partitions of the interval $[a, b]$,

$$\mathcal{P} = \{ P = \{x_1, \ldots, x_{n_P}\} \,|\, P \text{ is a partion of } [a,b] \text{ with } x_i \le x_{i+1} \text{ for } 1 \le i \le n_P \}.$$

If f is differentiable and its derivative is integrable, then

$$V_a^b(f) = \int_a^b |f'(x)| \mathrm{d}x.$$

Exercise 1.8 *Decide and justify whether the Fourier series of the following functions will converge. What happens at points of discontinuity?*

1. *The function* $f_1 : [-\pi, \pi] \to \mathbb{R}$ *with* $f_1(x) = x^4 - 2\pi^2 x^2$*, which is extended periodically to* $\mathbb{R}$*.*
2. *The sawtooth wave with period* 2π*:* $f_2 : \mathbb{R} \to \mathbb{R}$ *given by* $f_2(x) = 2\left(\frac{x}{\pi} - \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor\right)$*.*
3. *The triangle wave with period* 2π*:* $f_3 : \mathbb{R} \to \mathbb{R}$ *given by* $f_3(x) = \left|2\left(\frac{x}{\pi} - \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor\right)\right|$*.*
4. *The series* $f_4 : \mathbb{R} \to \mathbb{R}$ *given by* $f_4(x) = \sum_{n\in\mathbb{Z}} e^{-(x-2\pi n)^2}$*.*
5. *The tangent* $\tan : \cup_{k\in\mathbb{Z}}(\frac{2k-1}{2}\pi, \frac{2k+1}{2}\pi) \to \mathbb{R}$*.*

We get a stronger notion of converges if we Fourier coefficients are absolutely summable.

Proposition 1.9 *If the Fourier series converges pointwise to* f *and its Fourier coefficients are absolutely summable, then the Fourier series converges uniformly.*

Exercise 1.10 *Decide if the Fourier series in Exercise 1.8 also converge uniformly.*

Thus far we saw conditions on f that ensure convergence of its Fourier series. But what if we find a trigonometric representation of a periodic function, is it necessarily its Fourier series?

Theorem 1.11 *If a* 2π*-periodic function* f *is represented as a trigonometric series which converges uniformly on* $\mathbb{R}$*, then this series is the Fourier series of* f*.*

Proof Suppose that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$, where a_0, a_n, b_n are 'unknown', and that the series is uniformly convergent. As we have uniform convergence, term by term integration of the series is allowed. Thus, in analogy to Section 1.3 we get a_0 as in Equation (1.5). Next, we multiply both sides by $\cos(kx)$ and get

$$f(x)\cos(kx) = \frac{a_0}{2}\cos(kx) + \sum_{k=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)\cos(kx)\,.$$

We have to show that also this series converges uniformly. Since the Fourier series of f converges uniformly, we find an N_ϵ for any $\epsilon > 0$, such that for all $N \geq N_\epsilon$,

$$|f(x) - S_{f,N}(x)| \leq \epsilon\,.$$

Hence, we get for all $N \geq N_\epsilon$,

$$|f(x)\cos(kx) - S_{f,N}(x)\cos(kx)| = |f(x) - S_{f,N}(x)||\cos(kx)| \leq \epsilon\,,$$

and thus, this series can be integrated term by term, and we get a_n as in Equation (1.6). Similarly, we can derive the formula in Equation (1.7). □

Theorem 1.12 *If an absolutely integrable function* f *of period* 2π *can be expanded in a trigonometric series which converges to* f *everywhere, except possibly at a finite number of points (within one period), then this series is the Fourier series of* f*.*

Finally, we want to mention pointwise convergence almost everywhere (with respect to the Lebesgue measure). Functions in L^p are defined up to null sets, which means that two functions are identified if they only differ on a null set. We say that the two functions are equal *almost everywhere*. We say that a subset $X \in \mathbb{R}$ is a null set (a set with Lebesgue measure 0) if and only if for all $\epsilon > 0$ there is a sequence of intervals that contain X and the length of their union is less than ϵ.

Theorem 1.13 (Carleson-Hunt) *If f is a 2π-periodic function in $L^p([0,2\pi])$ for $p \in (1,\infty)$ and a_0, a_n, and b_n are defined as above, then for almost every $x \in \mathbb{R}$*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big).$$

Exercise 1.14 *Briefly discuss Exercise 1.8 using Theorem 1.13.*

1.5 General periodic function and complex Fourier coefficients

Until now we have considered functions of period 2π. But how can we represent functions of arbitrary period P? The next lemma gives us a simple way to change the period of a given periodic function.

Lemma 1.15 *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π. The function $g : \mathbb{R} \to \mathbb{R}$ defined point-wise by*

$$g(x) = f\Big(\frac{2\pi}{P} x\Big)$$

has period P.

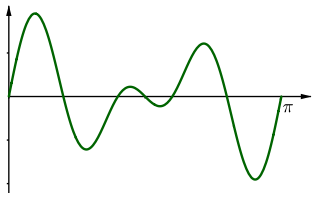
Proof Using periodicity of f, a short calculation yields the assertion,

$$g(x+P) = f\Big(\frac{2\pi}{P}(x+P)\Big) = f\Big(\frac{2\pi}{P}x + 2\pi\Big) = f\Big(\frac{2\pi}{P}x\Big) = g(x).$$

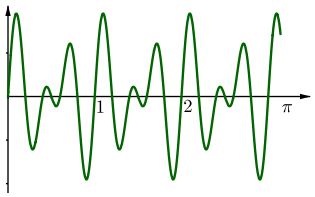
□

From Lemma 1.15 we get in particular that for a function f of period P the function g defined by $g(x) = f(Px)$ has period 1. So as a mnemonic we could remember that when multiplying the argument of a given function with its period, we 'squeeze' one cycle into an interval of length 1. From there we can divide the argument by any $P > 0$ to 'stretch' the cycle to an interval of length P.

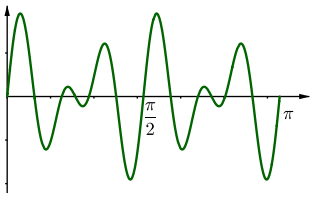
Example 1.16 1. Consider for example the periodic function from Figure 0.3. Let f denote the original function with period π. We can define a function g of period 1 by multiplying the argument with π as $g(x) = f(\pi x)$. Furthermore, we can divide the argument of g by $\frac{\pi}{2}$ as $h(x) = g(\frac{2x}{\pi}) = f(2x)$ to get a function of period $\frac{\pi}{2}$.



(1) Function f of period π



(2) Function g of period 1 defined by $g(x) = f(\pi x)$



(3) Function h of period $\frac{\pi}{2}$ defined by $h(x) = g(\frac{2x}{\pi})$

Figure 1.2: Visualisation of the procedure to change periods

2. Let us redefine the harmonic from (1.2) so that it has period 1. For this we multiply the argument with $\frac{2\pi}{\omega}$ and get

$$g(x) = A\sin\Big(\frac{2\pi}{\omega}\omega x + \frac{2\pi}{\omega}\varphi\Big) = A\sin(2\pi x + \hat{\varphi})\,.$$

3. We can modify the trigonometric sum in Equation (1.4) to get any period we want. For example, by Exercise 1.2 and Lemma 1.15, we get that

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\Big(a_n\cos(2\pi nx) + b_n\sin(2\pi nx)\Big) \tag{1.8}$$

is a function of period 1.

Exercise 1.17 *In this exercise, we want to generalise the coefficients* (1.5)*,* (1.6)*, and* (1.7) *derived in Section 1.3 for sums of arbitrary periods. Let* $\omega_0 = \frac{2\pi}{P}$ *and consider the sum*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\Big(a_n\cos(\omega_0 nx) + b_n\sin(\omega_0 nx)\Big).$$

By Exercise 1.2 and Lemma 1.15 we know that f *has period* P*. In analogy with the procedure in Section 1.3, derive the coefficients* a_0, a_n *and* b_n *for* $n \in \mathbb{N}$ *following these steps:*

1. *Consider Lemma 1.3 and make necessary changes to calculate the integrals.*
2. *Integrate* f *over the interval* $[0, P]$ *and show that*

$$a_0 = \frac{2}{P}\int_0^P f(x)\mathrm{d}x\,.$$

3. *Multiply* f *with* $\cos(\omega_0 nx)$ *and integrate over* $[0, P]$ *to show that*

$$a_n = \frac{2}{P}\int_0^P f(x)\cos(\omega_0 nx)\,\mathrm{d}x \quad \textit{for } n \in \{1,\dots,N\}\,.$$

4. *Multiply* f *with* $\sin(\omega_0 nx)$ *and integrate over* $[0, P]$ *to show that*

$$b_n = \frac{2}{P}\int_0^P f(x)\sin(\omega_0 nx)\,\mathrm{d}x, \quad \textit{for } n \in \{1,\dots,N\}\,.$$

Compare your results to Exercise 1.4.

In the following exercise, you will derive another version of the trigonometric sum using complex exponentials. This representation is simpler and more elegant to work with. You will often find the 'complex version' of the Fourier coefficient, so it is important that you feel comfortable with this version.

Exercise 1.18 *As you may know, the sine and the cosine can be written in terms of complex exponentials,*

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}\,, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}\,, \tag{1.9}$$

where i denotes the imaginary unit. Using this, we want to write the sum in Equation (1.8) *in the form*

$$f(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}. \tag{1.10}$$

In this exercise, you will derive this representation and show how the coefficients c_n relate to the coefficients a_0, a_n, and b_n derived in (1.5), (1.6), *and* (1.7), *respectively. Follow these two steps:*

1. *Start with Equation* (1.8) *and substitute the sines and cosines with the expressions from Equation* (1.9).
2. *Bracket your expression in pairs of $e^{-2\pi i n x}$ and $e^{2\pi i n x}$ and then change the limits of the sum such that there are only expressions $e^{2\pi i n x}$.*

In summary, you have just shown that

$$c_0 = \frac{a_0}{2}, \text{ and for } n \in \{1, \dots, N\} : c_n = \frac{a_n - i b_n}{2} \text{ and } c_{-n} = \frac{a_n + i b_n}{2}.$$

Note that $c_0 = \int_0^1 f(x)\,\mathrm{d}x$ is the average of the function f over the interval $[0,1]$. Also note that since the sum in Equation (1.8) *is real and $e^{-2\pi i n x}$ is the complex conjugate of $e^{2\pi i n x}$, we have $c_{-n} = \overline{c_n}$ for all $n \in \{1, \dots N\}$. This means, if you have found c_n in a form $\alpha + i\beta$ you can just change the sign of the imaginary part to get $c_{-n} = \alpha - i\beta$. Similarly, if you are given complex coefficients, you can get the real ones as $a_n = c_n + c_{-n} = 2\Re(c_n)$ and $b_n = i(c_n - c_{-n}) = -2\Im(c_n)$, for $n \in \{1, \dots, N\}$.*

Remark 1.19 It is common to denote the Fourier coefficients by $\widehat{f}(n) = c_n$.

1.6 Even and odd functions

One nice thing about even and odd functions is that we can simplify the corresponding Fourier series. Let f be an *even* function defined on the interval $[-\pi, \pi]$ (or an even periodic function on $\mathbb{R}$). Since $x \mapsto \cos(nx)$ is an even function for any $n \in \mathbb{N}$, Exercise 0.21 tells us that the function $x \mapsto f(x)\cos(nx)$ is also even for any $n \in \mathbb{N}$. On the other hand, the function $x \mapsto \sin(nx)$, $n \in \mathbb{N}$, is odd, so that the function $x \mapsto f(x)\sin(nx)$ is also odd. Then using (1.6), (1.7), (0.4) and (0.5), we find that the Fourier coefficients of the even function f are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \quad n \in \mathbb{N}_0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = 0, \quad n \in \mathbb{N}.$$

Therefore, the Fourier series of an even function contains only *cosines*,

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

Similarly, let f be an *odd* function, defined on the interval $[-\pi, \pi]$, or else an odd periodic function. Since $\cos(nx)$ is an even function, the function $x \mapsto f(x)\cos(nx)$ is odd, and since $\sin(nx)$ is odd, the function $f(x)\sin(nx)$ is even. Then using (1.6), (1.7), (0.4) and (0.5), we find that the Fourier coefficients of the odd function f are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x = 0 \quad n \in \mathbb{N}_0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad n \in \mathbb{N}.$$

Therefore, the Fourier series of an odd function contains only *sines*,

$$f \sim \sum_{n=1}^{\infty} b_n \sin(nx)\,.$$

Since the Fourier series of an odd function contains only sines, it vanishes for all points $x = k\pi, k \in \mathbb{Z}$, regardless of the values of f at these points.

Example 1.20 Let us consider the square wave $f : \cup_{k\in\mathbb{Z}}(k, k+1) \to \{-1, 1\}$ from Example 0.16 defined as

$$f(x) = \begin{cases} -1 & \text{if} \quad \mod(\lfloor 2x \rfloor, 2) = 0\,, \\ 1 & \text{else.} \end{cases}$$

Note that this function has period 1 and recall Exercise 1.17. Note that f is odd, that means for all $x \in \cup_{k\in\mathbb{Z}}(k, k+1)$ we have $f(x) = -f(-x)$. If we assume now that we can represent f as a trigonometric series, then we know from the discussion above that $a_n = 0$ for all $n \in \mathbb{N}_0$. A short calculation yields now

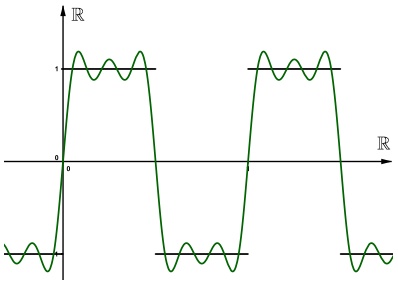
$$\begin{aligned} b_n &= 2\int_0^1 f(x)\sin(2\pi nx)\,\mathrm{d}x = 2\Big(\int_0^{\frac{1}{2}} f(x)\sin(2\pi nx)\,\mathrm{d}x + \int_{\frac{1}{2}}^1 f(x)\sin(2\pi nx)\,\mathrm{d}x\Big) \\ &= 2\Big(\int_0^{\frac{1}{2}} \sin(2\pi nx)\,\mathrm{d}x - \int_{\frac{1}{2}}^1 \sin(2\pi nx)\,\mathrm{d}x\Big) = \frac{1}{\pi n}\Big(-\cos(2\pi nx)\Big|_0^{\frac{1}{2}} + \cos(2\pi nx)\Big|_{\frac{1}{2}}^1\Big) \\ &= \frac{1}{\pi n}(2 - 2\cos(\pi n)) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd}\,. \end{cases} \end{aligned}$$

Therefore, the Fourier series becomes

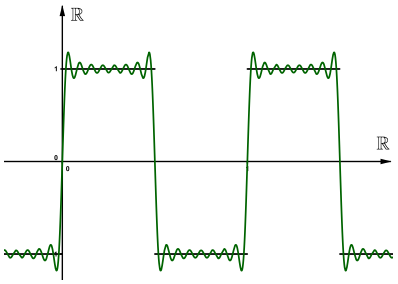
$$4\sum_{k=0}^{\infty} \frac{\sin(2\pi(2k+1)x)}{\pi(2k+1)}\,.$$

For practical applications we must of course cut this series at some point, since we simply cannot keep adding terms (on the computer). For most applications this is not a problem, since often the sum converges quickly and we accept a tiny (not significant) error. In Figure 1.3, we see the graphs of four approximations of the square wave with $N \in \{5, 15, 25, 99\}$.
Obviously, the approximations get better the bigger N becomes. Would you say the approximations are good? Are these approximations good enough that we can call the error 'not significant'? What do you notice at the jump discontinuities? This is called *Gibbs phenomenon*.

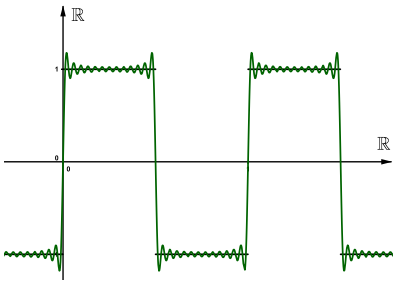
Note that if we define the square wave on whole $\mathbb{R}$, then it is not odd any more. Can you explain why? *Hint:* What value must an odd function take at 0? Does this have any practical effect on the Fourier series?



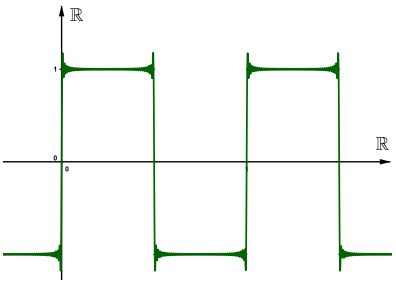
(1) Fourier sum with $N = 5$



(2) Fourier sum with $N = 15$



(3) Fourier sum with $N = 25$



(4) Fourier sum with $N = 99$

Figure 1.3: Approximation of the square wave with Fourier sums of different lengths

2 Systems of linear differential equations

In this section, we will see how system of linear differential equations can be approached. The structure of this section partly resembles the structure in the lecture notes of Annette A'Campo-Neuen.[9] We already know linear differential equations of the form $x'(t) = ax(t)$, for $x : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. And we also know that $x(t) = Ce^{at}$. This solution is quite straightforward. But in the real-world, many processes, which we describe by differential equations, depend on other processes, which can also be described by differential equations.

2.1 Systems of first order linear differential equations

This consideration naturally leads us to *coupled* differential equations. We call such a collection a *system of first order coupled linear differential equations*,

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + \ldots + a_{1n}(t)x_n(t) + b_1(t)\,, \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1(t) + \ldots + a_{nn}(t)x_n(t) + b_n(t)\,. \end{aligned} \tag{2.1}$$

The coefficients $a_{k\ell}$ and b_k can be constants in $\mathbb{R}$ or functions of t, and $x_1, \ldots, x_n$ are the wanted differentiable real-valued functions. For simplicity we will consider constant coefficients. If all b_k are equal 0, then we say the system is *homogeneous*, otherwise we call it *non-homogeneous*. At first, this system looks quite confusing. But imagine now that you are in $\mathbb{R}^n$ and that our n real-valued functions x_k are actually just a single vector-valued function $x : \mathbb{R} \to \mathbb{R}^n$, with $t \mapsto x(t) = (x_1(t), \ldots, x_n(t))^\intercal \in \mathbb{R}^n$. If all x_k are differentiable, then also x is (component-wise) differentiable and we write $x' = (x_1', \ldots, x_n')^\intercal$.
Let us take a further look at Equation (2.1). We notice that the expressions on the right hand side look like the vector $x(t)$ was multiplied by a matrix and then a vector was added. Indeed, if we write

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

we can express the system in Equation (2.1) simply as $x'(t) = Ax(t) + b$. In additional to the considerable simplification of notation we will see that this approach is a great help in solving systems of differential equations. For simplicity we will focus on homogeneous systems with constant coefficients, that is $A \in \mathbb{R}^{n\times n}$ and $b = 0$, which take the form

$$x'(t) = Ax(t)\,. \tag{2.2}$$

Notice that for any solutions x_1, x_2 of Equation (2.2) all linear combinations $a_1x_1 + a_2x_2$ are also solutions. Hence, the solution set of Equation (2.2) $X = \{x : \mathbb{R} \to \mathbb{R}^n \mid \forall t \in \mathbb{R} : x'(t) = Ax(t)\}$ is a *linear subspace* of the vector space $\mathcal{V}$ of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$. We will see in Theorem 2.5 that every initial value problem of the form (2.2) with $x(0) = x_0 \in \mathbb{R}^n$ has a unique solution in $\mathcal{V}$. With this in mind we get the following result.

Proposition 2.1 *Let $A \in \mathbb{R}^{n\times n}$ and $X = \{x : \mathbb{R} \to \mathbb{R}^n \mid \forall t \in \mathbb{R} : x'(t) = Ax(t)\}$. Then $\dim X = \dim \mathbb{R}^n = n$.*

[9] For additional ideas and many examples consider https://math.unibas.ch/fileadmin/mathe/redaktion/feupload/skript5_08.pdf and https://math.unibas.ch/fileadmin/mathe/redaktion/feupload/skript3_05.pdf.

Proof Consider the projection (in diesem Fall eine sogenannte *Auswertungsabbildung*) $X \to \mathbb{R}^n$ given as $x \mapsto x(0)$. This map is a linear surjection. By the uniqueness of solutions to initial value problems it is actually a bijection. Hence, X and $\mathbb{R}^n$ are isomorph and thus, $\dim X = \dim \mathbb{R}^n$. □

Let us try and solve Equation (2.2) for the simple case of a diagonal matrix A. We know that a diagonal matrix consists of its eigenvalues and we write $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Evaluating Equation (2.2), we get n (one-dimensional) differential equations of the form $x_k'(t) = \lambda_k x_k(t)$, $k \in \{1, \dots, n\}$. So the system is actually *uncoupled* and we already know the solutions

$$x(t) = (c_1 e^{\lambda_1 t}, \dots, c_n e^{\lambda_n t})^\intercal, \text{ for arbitrary constants } c_1, \dots, c_n \in \mathbb{R}.$$

In fact, we could choose n specific solutions

$$y_1(t) = (e^{\lambda_1 t}, 0, \dots, 0)^\intercal, \; y_2(t) = (0, e^{\lambda_2 t}, \dots, 0)^\intercal, \dots, \; y_n(t) = (0, \dots, 0, e^{\lambda_n t})^\intercal \in \mathbb{R}^n.$$

These are linearly independent, since $\det((y_1(t)|\cdots|y_n(t))) = e^{\lambda_1 t} \cdots e^{\lambda_n t} \neq 0$ for all $t \in \mathbb{R}$. So, by Proposition 2.1, $\{y_1, \dots, y_n\}$ forms a basis of X. Indeed, we quickly realise that we can write every solution as a linear combination of all y_k as $x = c_1 y_1 + \ldots + c_n y_n$ and that if $x = 0$, then all c_k must be 0.
We call such a basis a *fundamental system of solutions (Fundamentalsystem der Differentialgleichung)*. Furthermore, we note that the y_k are all of the form $e^{\lambda_k t} v_k$, where v_k are the eigenvectors (in this case the standard basis of $\mathbb{R}^n$) of A corresponding to the eigenvalues λ_k. It is natural to ask if this is true for arbitrary matrices. Luckily, we can affirm this.

Proposition 2.2 *Let $v \in \mathbb{R}^n$ be an eigenvector of a matrix $A \in \mathbb{R}^{n \times n}$ corresponding to the eigenvalue λ. Then $x(t) = e^{\lambda t} v$ satisfies Equation* (2.2).

Proof A short calculation yields

$$Ax(t) = A(e^{\lambda t} v) = e^{\lambda t} Av = e^{\lambda t} \lambda v = \frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda t} v) = x'(t),$$

and hence, x is a solution. □

Exercise 2.3 *Intuitively, the next best thing in this context would probably be diagonalisable matrices. Explain what we mean by this and derive a fundamental system of solutions of* (2.2) *given a diagonalisable matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$. Use Proposition 2.2 and argue that $e^{\lambda_1 t} v_1, \dots, e^{\lambda_n t} v_n$ are linearly independent. Maybe recall Section 0.4.5.*

We will discuss these new concepts on a practical example.

Exercise 2.4 *In this exercise, we want to find a fundamental system of solutions to the following linear homogeneous system of first order differential equations,*

$$\begin{aligned} x_1'(t) &= -x_1(t) + 2x_2(t), \\ x_2'(t) &= -3x_1(t) + 4x_2(t). \end{aligned} \tag{2.3}$$

By following the steps below, we revise many concepts we have learned so far, for example in this section and Section 0.4.4.

1. Set $x = (x_1, x_2)^\intercal$ and find a (2×2)-matrix A to express (2.3) *in the form $x'(t) = Ax(t)$.*

2. *Write down the characteristic polynomial and calculate its roots to find the eigenvalues of A. You should get $\lambda_1 = 1$ and $\lambda_2 = 2$.*
3. *Derive the corresponding eigenvectors and briefly argue that they are linearly independent. The eigenvectors should be multiples of $v_1 = (1,1)^\intercal$ and $v_2 = (2,3)^\intercal$.*
4. *Write down the fundamental system of solutions according to Exercise 2.3. Check that indeed every linear combination of this fundamental system solves Equation* (2.3)*.*

Having covered diagonal and diagonalisable matrices, we move on to arbitrary matrices. As promised in Section 0.4.6, we will now prove a very useful application of the matrix exponential.

Theorem 2.5 *Let $A \in \mathbb{R}^{n\times n}$ and $x_0 \in \mathbb{R}^n$ be given. The system of homogeneous linear first order differential equations* (2.2) *with the initial condition $x(0) = x_0$ yields a unique solution $x : \mathbb{R} \to \mathbb{R}^n$ given by*

$$x(t) = e^{tA}x_0\,, \quad \text{for all } t \in \mathbb{R}\,.$$

The columns of e^{tA} form a fundamental system of solutions of (2.2)*.*

Proof Let $x(t) = e^{tA}x_0$ and recall that we derived $\frac{\mathrm{d}}{\mathrm{d}t}e^{tA} = Ae^{tA}$ in Equation (0.8). This implies $x'(t) = Ax(t)$. The initial condition is also satisfied, since $x(0) = \mathbb{1}_n x_0 = x_0$.
To show uniqueness, we assume that y is an arbitrary solution of (2.2) with $y(0) = x_0$. Consider the map $c : \mathbb{R} \to \mathbb{R}^n$ with $t \mapsto e^{-tA}y(t)$ and calculate its derivative. We find that

$$c'(t) = -Ae^{-tA}y(t) + e^{-tA}y'(t) = -Ae^{-tA}y(t) + e^{-tA}Ay(t) = 0\,.$$

The last equality holds, since A and e^{-tA} commute. Hence, the function c is constant, and since y satisfies $y(0) = x_0$, we have for all $t \in \mathbb{R}$

$$c(t) = c(0) = \mathbb{1}_n x_0 = x_0\,,$$

proving that in fact, $y = x$.
For the last assertion we notice that the columns of e^{tA}, let us denote them by $a_1, \ldots, a_n$, are linearly independent, since by Proposition 0.37, $\det(e^{tA}) \neq 0$. Multiplying the k-th standard basis vector $e_k = (0, \ldots 0, 1, 0 \ldots, 0)^\intercal$ by e^{tA} we get its k-th column, $e^{tA}e_k = a_k$. Hence, a_k is a solution to (2.2) with the initial condition $a_k(0) = e_k$, proving the assertion. □

Exercise 2.6 *We left some parts of the proof of Theorem 2.5 incomplete. Convince yourself that the following are correct.*

1. Show that e^{-tA} and A commute, recalling that we already know that e^{tA} and A commute.
2. Show that $\frac{\mathrm{d}}{\mathrm{d}t}(e^{tA}x_0) = \frac{\mathrm{d}}{\mathrm{d}t}(e^{tA})x_0$. How does the differential operator on the left hand side differ from the one on the right hand side?

Exercise 2.7 *We still need to show Assertion 3 of Proposition 0.37. We want to show that if two matrices $A, B \in \mathbb{R}^{n\times n}$ commute, then $e^{A+B} = e^A e^B$.*

1. First of all, we need an auxiliary results: we need to show that if A and B commute, then B and e^{tA} commute. Define the vector-valued function $x : \mathbb{R} \to \mathbb{R}^n$ by $x(t) = Be^{tA}c$, for an arbitrary $c \in \mathbb{R}^n$. Its derivative is given by $x'(t) = BAe^{tA}c$. Since $AB = BA$, we have $x'(t) = ABe^{tA}c = Ax(t)$. But Theorem 2.5 tells us that x is uniquely given by $x(t) = e^{tA}x(0) = e^{tA}Bc$. Hence, we must have $Be^{tA} = e^{tA}B$, for all $t \in \mathbb{R}$.

2. To show $e^{A+B} = e^A e^B$, define the function $x(t) = e^{tA}e^{tB}c$, for an arbitrary $c \in \mathbb{R}^n$. Proceed as in the first part and use the result derived in the first part.

Example 2.8 We want to calculate the exponential of the (2×2)-matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since this matrix is not diagonalisable (you can check this with Proposition 0.34), we cannot use the procedure outlined in Section 0.4.5 to calculate the powers of A. However, in simple cases like this we can sometimes 'guess' the solution. Let us calculate a few powers and see if we find a pattern,

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

It seems that $A^k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$, for $k \in \mathbb{N}$. We can prove this by induction. We have already established the induction start. The induction step goes as follows,

$$A^{k+1} = AA^k = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}.$$

With this it is straightforward to calculate e^{tA},

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & \sum_{k=0}^{\infty} \frac{kt^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!} \end{pmatrix} = \begin{pmatrix} e^t & t\sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & t\sum_{k=0}^{\infty} \frac{t^k}{k!} \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}.$$

Exercise 2.9 *Calculate the matrix exponentials of*

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ and } C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \text{ for } \lambda, a, b \in \mathbb{R}.$$

Find the fundamental systems of solutions for $x'(t) = Bx(t)$ and $x'(t) = Cx(t)$.
Hint: *Show by induction that $B^k = \left(\begin{smallmatrix} 1 & k\lambda^{k-1} \\ 0 & 1 \end{smallmatrix}\right)$ and use the procedure outlined in Section 0.4.5 to calculate C^k.*

Let us briefly discuss one more useful way to calculate matrix exponentials. As discussed in Section 0.4.5, we can quite easily calculate powers of a diagonalisable matrix A. Let P and P^{-1} be such that $P^{-1}AP = D$ is diagonal. Since the matrix exponential is a series of powers, it might not be surprising that

$$e^A = e^{PDP^{-1}} = \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} = P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)P^{-1} = Pe^DP^{-1}.$$

The exponential e^D is easily computed. If you calculated e^{tC} in Exercise 2.9, you will have used this fact. However, not all matrices are diagonalisable.
It turns out that the diagonal form of a diagonalisable matrix is just a special case of what is known as the *Jordan normal form (Jordansche Normalform)*. Every matrix whose eigenvalues are contained in the field over which the vector space is defined exhibits such a form. For example, for every matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues there exists an invertible matrix P such that $P^{-1}AP$ takes the form

$$P^{-1}AP = J = J_{\lambda_1} \oplus \ldots \oplus J_{\lambda_r} = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_r} \end{pmatrix},$$

where $\oplus$ denotes the *direct sum* of two matrices and each J_{λ_k} is called a *Jordan block corresponding to λ_k* and is a square matrix of the form

$$J_{\lambda_k} = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}.$$

The number of Jordan blocks corresponding to a given eigenvalue λ_k is equal to the *geometric multiplicity* of λ_k and the sum of sizes of all Jordan blocks corresponding to λ_k equals its *algebraic multiplicity*. We strongly encourage you to read some additional literature on the Jordan normal form, as we cannot go into much detail here. The point we want to make is the following. Just as in the diagonalisable case, we have for a Jordan normal form $P^{-1}AP = J$,

$$e^A = Pe^J P^{-1}.$$

It turns out that e^J is the direct sum of the exponential of each block, $e^J = e^{J_{\lambda_1}} \oplus \ldots \oplus e^{J_{\lambda_r}}$. Now every Jordan block is of the form $J_k = \lambda_k \mathbb{1} + N$, where N is a nilpotent matrix of the form

$$N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

this means there exists an $n \in \mathbb{N}$ such that $N^j = 0$ for all $j \geq n$. With a short consideration we convince ourselves that $\lambda_k \mathbb{1}$ and N commute and thus, by Exercise 2.7, $e^{\lambda_k \mathbb{1} + N} = e^{\lambda_k \mathbb{1}} e^N$. Now $e^{\lambda_k \mathbb{1}}$ is readily calculated, and since N is nilpotent, we get $e^N = \sum_{k=0}^{\infty} \frac{N^k}{k!} = \sum_{k=0}^{n-1} \frac{N^k}{k!}$ is just a sum of powers.

Let us consider an example.

Example 2.10 Let $B = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ from Exercise 2.9 with $\lambda \in R$. We already know that B is not diagonalisable. However, it has Jordan normal form (with a single Jordan block) and we can write $B = \lambda \mathbb{1}_2 + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \lambda \mathbb{1}_2 + N$. Notice that $N^k = 0$, for $k \geq 2$. Hence,

$$e^{tB} = e^{t\lambda \mathbb{1}_2} e^{tN} = e^{t\lambda} \mathbb{1}_2 \left((tN)^0 + (tN)^1\right) = e^{t\lambda}(\mathbb{1}_2 + tN) = e^{t\lambda} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

the solution you should have derived in Exercise 2.9.

Exercise 2.11 *Consider the following two matrices,*

$$A = \begin{pmatrix} -3 & -2 & -6 & -5 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 6 & 9 \\ -1 & 0 & -1 & -4 \end{pmatrix} \quad \textit{and} \quad P = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We picked them nicely so that $P^{-1}AP = J$ is the Jordan normal form of A. Calculate e^A.
Hint: *For this you will calculate P^{-1}, J, e^J, and finally e^A. To calculate e^J you will use that e^J is the direct sum of the exponential of each block, $e^J = e^{J_{\lambda_1}} \oplus \ldots \oplus e^{J_{\lambda_r}}$.*

2.2 Linear multi-compartment models

In this section, we will see how systems of linear differential equations arise. We will focus on the derivation of the systems and in particular on the derivation of the corresponding matrix.
For more material, consider also the slides (Kapitel 3 - Modelle).

In the following, we will discuss certain linear interactions between multiple compartments. We will assume that each compartment has an assigned quantity of some drug or something similar. The interactions are assumed to take place at constant rates, and thus, we will use indexed constants

to represent these. Let us agree to use the first index to denote the number of the compartment to which we add the quantity and the second index to denote the number of the compartment from which the quantity is subtracted. Consider the following short example.

Example 2.12 Let $y_1(t)$ and $y_2(t)$ be the quantities of a drug in organs 1 and 2 at time t. We assume that the drug is introduced to Organ 1 at a rate $M(t)$, where it is degraded at a constant rate a_1 and also passed on to Organ 2 at a rate b_{12}. In Organ 2, the drug is degraded at rate a_2 and passed back to Organ 2 at a rate b_{21}. We can visualise these relationships in the diagram in Figure 2.1.

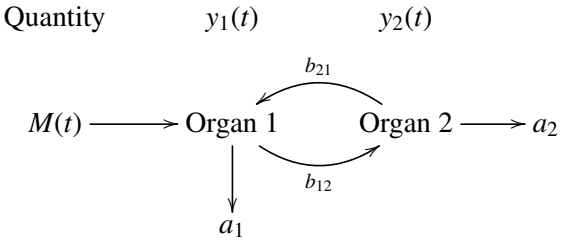


Figure 2.1: Example of a linear multi-compartment model.

From Figure 2.1 we can directly derive the differential equations,

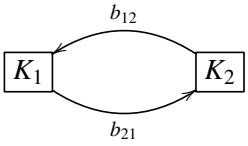
$$y_1'(t) = M(t) - (a_1 + b_{12})y_1(t) + b_{21}y_2(t),$$
$$y_2'(t) = b_{12}y_1(t) - (b_{21} + a_2)y_2(t).$$

Hence, defining for every t, $y = (y_1, y_2)^\intercal$, $y' = (y_1', y_2')^\intercal$, $g = (M, 0)^\intercal$, and

$$A = \begin{pmatrix} -(a_1 + b_{12}) & b_{21} \\ b_{12} & -(b_{21} + a_2) \end{pmatrix},$$

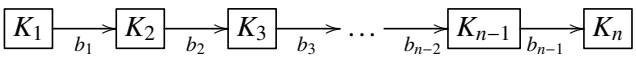
we can compactly express this model as $y'(t) = Ay(t) + g(t)$. With the tools from previous sections, you can already solve this for $M = 0$.

Exercise 2.13 *Consider the following 2-compartment model. Assume that the corresponding quantity of a drug in compartment K_j at time t is $y_j(t)$, for $j \in \{1, 2\}$.*



Write down the system of differential equations which represents this model in matrix notation. Solve this system for $b_{12} = b_{21} = 1$.

Example 2.14 Let us consider a special compartment model in the form



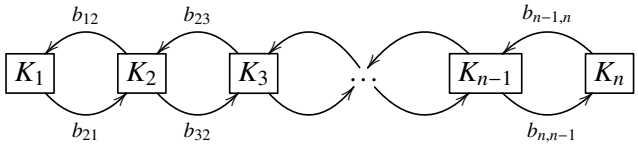
The system is given by

$$y_1' = -b_1 y_1\,,\ \ y_2' = b_1 y_1 - b_2 y_2\,,\ \ \ldots,\ y_n' = b_{n-1} y_{n-1}\,.$$

Convince yourself that the system can be written as $y'(t) = Ay(t)$ with

$$A = \begin{pmatrix} -b_1 & & & & \\ b_1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & -b_{n-1} & \\ & & & b_{n-1} & 0 \end{pmatrix}.$$

Exercise 2.15 *Consider the following model.*



Derive the system of differential equations and derive the corresponding matrix.

2.3 Linear differential equations of n-th order

In this section, we will consider a single differential equation of n-th order. Interestingly, we can translate this equation to a system of first order equations. Similarly to the previous section, we are able to solve this system. Consider the following n-th order homogeneous linear differential equation,

$$x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1 x' + a_0 x = 0\,. \tag{2.4}$$

To generate n first order differential equations we define the vector-valued function $y : \mathbb{R} \to \mathbb{R}^n$ as $y = (x, x', \ldots, x^{(n-1)})^\intercal$. Convince yourself that if x satisfies Equation (2.4), then y is a solution to the following system

$$\begin{aligned} y_1'(t) &= y_2(t)\,, \\ &\vdots \\ y_{n-1}'(t) &= y_n(t)\,, \\ y_n'(t) &= -a_0 y_1(t) - a_1 y_2(t) - \ldots - a_{n-1} y_n(t)\,. \end{aligned}$$

Notice that the last equality is exactly Equation (2.4). We can now summarise the coefficients of this system in a matrix

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix} \in \mathbb{R}^{n\times n},$$

so that the system takes the form

$$y'(t) = Ay(t)\,.$$

In analogy to Section 2.1, we need to find the eigenvalues of A. So we want to derive the characteristic polynomial of A. We will find that it is given by

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0\,. \tag{2.5}$$

To show this, we can use induction over n.
For $n = 1$, the differential equation reads $x^{(1)} + a_0 x = 0$ and thus, we have $A = -a_0 \in \mathbb{R}$. Hence,

$$p_A(\lambda) = \det(\lambda - (-a_0)) = \lambda + a_0 \,. \qquad \checkmark$$

For the induction step $n \to n+1$, assume that Equation (2.5) holds true. We use the Laplace formula with respect to the first column and get

$$\begin{aligned} p_A(\lambda) = \det(\lambda \mathbb{1}_{n+1} - A) &= \begin{vmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \\ a_0 & \cdots & a_{n-1} & \lambda + a_n \end{vmatrix} \\ &= (-1)^2 \lambda \cdot \begin{vmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \\ a_0 & \cdots & a_{n-1} & \lambda + a_n \end{vmatrix} + (-1)^{n+2} a_0 \cdot \begin{vmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \\ a_0 & \cdots & a_{n-1} & \lambda + a_n \end{vmatrix} \\ &= \lambda(\lambda^n + a_n \lambda^{n-1} + \ldots + a_2 \lambda + a_1) + (-1)^{n+2+n} a_0 = \lambda^{n+1} + a_n \lambda^n + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 \,. \end{aligned}$$

Notice that the characteristic polynomial (2.5) resembles the differential equation (2.4). So we call the polynomial the *characteristic polynomial of the differential equation* and we call A (or more often its transpose) the *Frobenius companion matrix* to that polynomial.
In analogy to the previous section, we know that the solution set is an n-dimensional vector space, so the same is true for the solution set of (2.4). We adopt the same terminology and call a set of functions $\{y_1, \ldots, y_n\}$ a fundamental system of solutions, if

$$Y_1 = \left(y_1, y_1', \ldots, y_1^{(n-1)}\right)^\mathsf{T}, \ldots, Y_n = \left(y_n, y_n', \ldots, y_n^{(n-1)}\right)^\mathsf{T}$$

is a basis of the solution set of the corresponding system of first order differential equations. To check linear independence, we calculate what is known as the *Wronskian (Wronski-Determinante)*,

$$W_{Y_1,\ldots,Y_n}(t) = \det((Y_1|\cdots|Y_n)) = \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} .$$

To find solutions, we have a very useful result.

Theorem 2.16 *Let p be the characteristic polynomial in* (2.5) *with real roots $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ and multiplicity $m_1, \ldots, m_r \in \mathbb{N}$. Then for $j \in \{1, \ldots, r\}$ and $k \in \{0, \ldots, m_k - 1\}$ the functions $t^k e^{\lambda_j t}$ form a fundamental system of solutions of* (2.4).
If the set of roots exhibits a pair of complex conjugate numbers $\lambda = a + ib$ and $\bar{\lambda} = a - ib$ with multiplicity m, then for $k \in \{0, \ldots, m-1\}$ the functions $t^k e^{at} \cos(bt)$ and $t^k e^{at} \sin(bt)$ are linearly independent solutions of (2.4).

Example 2.17 Consider the differential equation

$$x^{(3)} - 2x^{(2)} - x^{(1)} + 2x = 0 \,.$$

Its characteristic polynomial is given by $p(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda + 1)(\lambda - 2)$. Hence, the roots are given by $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$ and thus, according to Theorem 2.16, a fundamental system of solutions is given by the collection

$$x_1(t) = e^t \,, \quad x_2(t) = e^{-t} \,, \text{ and } \quad x_3(t) = e^{2t} \,.$$

Indeed, in this case the corresponding solutions

$$X_1 = \begin{pmatrix} x_1 \\ x_1' \\ x_1'' \end{pmatrix} = e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad X_2 = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ and } \quad X_3 = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

are the eigenvectors of the companion matrix $A = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{smallmatrix} \right)$.

Exercise 2.18 *Derive fundamental systems of solutions of the following differential equations.*

$$x^{(3)} - 5x^{(2)} + 3x^{(1)} + 9x = 0, \qquad x^{(2)} + x = 0.$$

Write down and factorise the characteristic polynomials to find its roots and then use Theorem 2.16.

3 Laplace Transformation

The Laplace transformation is an important and powerful tool in mathematics. Its applications reach from image and signal processing to solving partial differential equations. In this section, we introduce the Laplace transformation and its most important properties. At the end of this section, we will see how it helps us to solve differential equations.

3.1 Definition and Examples

Definition 3.1 The *Laplace transform* of a function $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is a function $\mathcal{L}f : \mathbb{C} \to \mathbb{R}$ and is defined by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,. \tag{3.1}$$

The symbol $\mathcal{L}$ denotes the Laplace operator which takes an $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ and maps it to its Laplace transform $\mathcal{L}f$.[10] Most common notations for the Laplace transform are $\mathcal{L}f = \mathcal{L}\{f\} = F$. If we want to calculate the Laplace transform of a formula (without name) we write $\mathcal{L}(f(t))$. If we consider the Laplace transform point-wise we write $\mathcal{L}f(s) = \mathcal{L}\{f\}(s) = \mathcal{L}(f(t))(s) = F(s)$, for $s \in \mathbb{C}$.

As you might imagine, using an arbitrary F to denote the Laplace transform of f can be confusing.[11] However, if you want to use the notation F rather then $\mathcal{L}f$ (which is exactly the same function) it is common to use the *Doetsch-symbol* to indicate the Laplace transform of the function f:

$$f \circ\!\!-\!\!\bullet\, F.$$

This sort of connection between to functions is sometimes called *correspondence*. It is particularly useful when we simply want to indicate the correspondence between two formulae, as in

$$t \circ\!\!-\!\!\bullet\, \frac{1}{s^2}\,,$$

meaning that the Laplace transform of $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $t \mapsto t$ is $\mathcal{L}f(s) = \frac{1}{s^2}$. Nevertheless, we encourage you to define your functions, stating domain and co-domain.

Let us define sufficient conditions for the existence of a Laplace transform.

Definition 3.2 We define the set E of all functions $f : \mathbb{R} \to \mathbb{C}$ such that

1. $f = 0$ on $\mathbb{R}_{<0}$,
2. $|f(t)| \leq Ce^{\alpha t}$ for some constants $\alpha \in \mathbb{R}$, $C \in \mathbb{R}_{>0}$, and all $t \in \mathbb{R}$,
3. f is piecewise continuous.

The first condition allows us to restrict the domain of f to $\mathbb{R}_{\geq 0}$. This is necessary in Definition 3.1 and it is sometimes useful to have functions on $\mathbb{R}$ rather than $\mathbb{R}_{\geq 0}$. Conversely, if we consider $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$, then we implicitly set $f = 0$ on $\mathbb{R}_{<0}$ to get an f defined on $\mathbb{R}$. For $f : \mathbb{R} \to \mathbb{R}$ which are not 0 on $\mathbb{R}_{<0}$, we just set $f = 0$ on $\mathbb{R}_{<0}$, as in the example above with $t \mapsto t$.

[10] In German for example, we actually differentiate between the function $F = \mathcal{L}f$ which is called 'Laplace-Transformierte der Funktion f' and $\mathcal{L}$ which is called 'Laplace-Transformation'. In English, both of them are called 'Laplace transform'. This is why we will call $\mathcal{L}$ the Laplace operator.

[11] Often F denotes the primitive integral of f, that is $F' = f$, which has nothing to do with Laplace transforms.

If f satisfies the second condition, we say '*f is of exponential order*', meaning that the rate of growth is at most that of exponential functions. Furthermore, we define the smallest such α as

$$\alpha_f = \inf\{\alpha \in \mathbb{R} \,|\, \forall t \in \mathbb{R}_{\geq 0} : |f(t)| \leq Ce^{\alpha t}\}.$$

The last conditions states that f is only allowed to have removable and jump discontinuities, no infinite or essential discontinuities are allowed. The following result shows that these conditions are sufficient for the existence of the Laplace transform.

Theorem 3.3 *Let $f \in E$, then $\mathcal{L}f$ is well-defined on $\{s \in \mathbb{C} | \Re(s) > \alpha_f\}$. Furthermore, we have $\lim_{\Re(s)\to\infty} \mathcal{L}f(s) = 0$.*

Proof Let $s_0 = x_0 + iy_0 \in \mathbb{C}$ such that $\Re(s_0) = x_0 > \alpha_f$. In particular there is an $\alpha < x_0$ and a $C > 0$ such that Condition 2 in Definition 3.2 is satisfied. Using that for any $y \in \mathbb{R}$ we have $|e^{iy}| = 1$, we get

$$|e^{-s_0 t} f(t)| = |e^{-(x_0+iy_0)t}||f(t)| = e^{-x_0 t}|f(t)| \leq Ce^{(\alpha - x_0)t}.$$

Therefore,

$$|\mathcal{L}f(s_0)| = \left| \int_0^\infty e^{-s_0 t} f(t)\,\mathrm{d}t \right| \leq \int_0^\infty |e^{-s_0 t} f(t)|\,\mathrm{d}t \leq \int_0^\infty Ce^{(\alpha - x_0)t}\,\mathrm{d}t = \frac{C}{x_0 - \alpha} < \infty,$$

and in particular, $|\mathcal{L}f(s_0)| \leq \frac{C}{x_0 - \alpha} \overset{x_0\to\infty}{\longrightarrow} 0$. □

Example 3.4 We consider a few short examples to get a feeling how this transform works. In all examples we directly calculate the Laplace transforms using Definition 3.1.

1. Let us find the Laplace transform of the *Heaviside step function* $\Theta : \mathbb{R} \to \{0, 1\}$ defined as

$$\Theta(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0. \end{cases}$$

First of all, we need to check that $\Theta \in E$. By definition $\Theta = 0$ on $\mathbb{R}_{<0}$ and has only one jump at $t = 0$. It is of exponential order with $\alpha_f = 0$ and $C = 1$. Hence, for any $s \in \mathbb{C}$ with $\Re(s) > 0$ the Laplace transform of Θ is well-defined and given by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st}\Theta(t)\,\mathrm{d}t = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{-s} e^{-st}\Big|_{t=0}^\infty = 0 - \frac{1}{-s} = \frac{1}{s}.$$

2. Consider the identity $\mathrm{id} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\mathrm{id}(t) = t$. As an exercise, explain how Condition 1 and Condition 3 in Definition 3.2 are satisfied. Show that also Condition 2 is satisfied with $\alpha_f = 0$. Therefore, the Laplace transform is defined for $s \in \mathbb{C}$ with $\Re(s) > 0$ and given by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} t\,\mathrm{d}t = \underbrace{-\frac{e^{-st}t}{s}\Big|_{t=0}^\infty}_{=0} + \int_0^\infty \frac{e^{-st}}{s}\,\mathrm{d}t = -\frac{e^{-st}}{s^2}\Big|_0^\infty = \frac{1}{s^2}.$$

In the second equality, we used integration by parts. Also note that by L'Hôpital's rule we have

$$\lim_{t\to\infty} |e^{-st}t| = \lim_{t\to\infty} \frac{t}{e^{\Re(s)t}} = \lim_{t\to\infty} \frac{1}{\Re(s)e^{\Re(s)t}} = 0.$$

3. Next we find the Laplace transform of $f : \mathbb{R}_{\geq 0} \to [-1,1]$ given by $f(t) = \sin(at)$, with $a > 0$. Show that $f \in E$ and that $\alpha_f = 0$. Then for any $s \in \mathbb{C}$ with $\Re(s) > 0$ we have

$$\begin{aligned}\mathcal{L}f(s) &= \int_0^\infty e^{-st}\sin(at)\,\mathrm{d}t = \frac{1}{2i}\int_0^\infty e^{-st}(e^{iat} - e^{-iat})\,\mathrm{d}t = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)\\ &= \frac{1}{2i}\left(\frac{s+ia}{s^2+a^2} - \frac{s-ia}{s^2+a^2}\right) = \frac{a}{s^2+a^2}\,.\end{aligned}$$

Note that we can also use integration by parts twice as

$$\begin{aligned}\int_0^\infty e^{-st}\sin(at)\,\mathrm{d}t &= \underbrace{-\frac{e^{-st}}{a}\cos(at)\Big|_{t=0}^{\infty}}_{=\frac{1}{a}} - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)\,\mathrm{d}t\\ &= \frac{1}{a} - \frac{s}{a}\Big(\underbrace{\frac{e^{-st}}{a}\sin(at)\Big|_{t=0}^{\infty}}_{=0} + \frac{s}{a}\int_0^\infty e^{-st}\sin(at)\,\mathrm{d}t\Big) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin(at)\,\mathrm{d}t\,.\end{aligned}$$

Then solving this equation for the integral we get

$$\int_0^\infty e^{-st}\sin(at)\,\mathrm{d}t = \frac{a}{s^2+a^2}\,.$$

Exercise 3.5 *Now you can practice at some more examples. Show that the following functions are contained in E and calculate their Laplace transforms stating their domains.*

1. *Let $f_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $f_1(t) = 1_{\{a\leq t\leq b\}}(t)$, where $0 \leq a \leq b$, and show that $\mathcal{L}f_1(s) = \frac{e^{-sa}-e^{-sb}}{s}$.*
2. *Let $f_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the n-th power $f_2(t) = t^n$, where $n \in \mathbb{N}$, and show that $\mathcal{L}f_2(s) = \frac{n!}{s^{n+1}}$.*
3. *Let $f_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the cosine $f_3(t) = \cos(at)$, with $a > 0$, and show that $\mathcal{L}f_3(s) = \frac{s}{s^2+a^2}$.*
4. *Let $f_4 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the exponential function $f_4(t) = e^{at}$, and show that for $s > a$, $\mathcal{L}f_4(s) = \frac{1}{s-a}$.*

3.2 Elementary properties of the Laplace Transformation

The Laplace transform has many interesting and useful properties. We will prove the most important ones and consider some examples. The following results are valid for $s \in \mathbb{C}$ with sufficiently large real part. Except for the linearity property below, we will omit giving the half-plane domain.

Proposition 3.6 (Linearity) *The Laplace operator $\mathcal{L}$ is complex-linear. This means that for $f, g \in E$ with Laplace transforms $\mathcal{L}f$, $\mathcal{L}g$ and two constants $a, b \in \mathbb{C}$ we have $af + bg \in E$ with*

$$\mathcal{L}\{af + bg\} = a\mathcal{L}f + b\mathcal{L}g\,.$$

Proof If f and g are 0 on $\mathbb{R}_{<0}$, then so is $af + bg$. The same holds true for piecewise continuity on $\mathbb{R}$. If $|f(t)| \leq C_1 e^{\alpha_1 t}$ and $|g(t)| \leq C_2 e^{\alpha_2 t}$ for all $t \in \mathbb{R}$, then

$$|af(t) + bg(t)| \leq |a||f(t)| + |b||g(t)| \leq Ce^{\alpha t}\,,$$

for $C = 2\max\{|a|C_1, |b|C_2\}$ and $\alpha = \max\{\alpha_1, \alpha_2\}$. Thus, $af + bg \in E$. Using Definition 3.1 and linearity of the integral, we get for $s \in \mathbb{C}$ with $\Re(s) > \max\{\alpha_{f_1}, \alpha_{f_2}\}$

$$\begin{aligned}\mathcal{L}\{a_1f_1 + a_2f_2\}(s) &= \int_0^\infty e^{-st}(a_1f_1(t) + a_2f_2(t))\,\mathrm{d}t\\ &= a_1\int_0^\infty e^{-st}f_1(t)\,\mathrm{d}t + a_2\int_0^\infty e^{-st}f_2(t)\,\mathrm{d}t\\ &= a_1\mathcal{L}f_1(s) + a_2\mathcal{L}f_2(s)\,,\end{aligned}$$

proving the assertion. □

Example 3.7 Consider the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $f(t) = 3t - 5t^2 + 3\sin(t)$. The results in Example 3.4 and Exercise 3.5 and the linearity property give us for $s \in \mathbb{C}$ with $\Re(s) > 0$

$$\mathcal{L}f(s) = 3\mathcal{L}(t)(s) - 5\mathcal{L}(t^2)(s) + 3\mathcal{L}(\sin(t))(s) = 3\frac{1}{s^2} - 5\frac{2}{s^3} + 3\frac{1}{1+s^2} = \frac{3}{s^2} - \frac{10}{s^3} + \frac{3}{1+s^2}.$$

Proposition 3.8 (Time scaling) *Consider $f \in E$ and an $a > 0$. Then the function $\mathbb{R}_{\geq 0} \ni t \mapsto f(at) \in \mathbb{R}$ belongs to E and has the Laplace transform*

$$\mathcal{L}(f(at))(s) = \frac{1}{a}\mathcal{L}(f(t))\left(\frac{s}{a}\right).$$

Proof A few considerations show that this function belongs to E. Then using the substitution $\hat{t} = at$ we get

$$\mathcal{L}(f(at))(s) = \int_0^\infty e^{-st} f(at)\,\mathrm{d}t = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}\hat{t}} f(\hat{t})\,\mathrm{d}\hat{t} = \frac{1}{a}\mathcal{L}f\left(\frac{s}{a}\right),$$

proving the assertion. □

If $0 < a < 1$, then we speak of *dilation*. In case $a > 1$, we speak of *contraction*.

Example 3.9 Let us calculate the Laplace transform of the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $f(t) = \sin(3t)$. According to the time scaling property, we have $\mathcal{L}(\sin(3t))(s) = \frac{1}{3}\mathcal{L}(\sin(t))\left(\frac{s}{3}\right)$. From Example 3.4 we already know that $\mathcal{L}(\sin(t))(s) = \frac{1}{1+s^2}$, and thus,

$$\mathcal{L}(\sin(3t))(s) = \frac{1}{3}\frac{1}{1+\left(\frac{s}{3}\right)^2} = \frac{3}{9+s^2}.$$

For which $s \in \mathbb{C}$ is this transform well-defined?

Proposition 3.10 (Shifting) *Let $f \in E$ and $h > 0$. The following results hold:*

1. Shifting to the right: $\mathcal{L}(f(t-h))(s) = e^{-hs}\mathcal{L}f(s)$,
2. Shifting to the left: $\mathcal{L}(f(t+h))(s) = e^{hs}\Big(\mathcal{L}f(s) - \int_0^h e^{-st} f(t)\,\mathrm{d}t\Big)$.

Proof As an exercise argue that $t \mapsto f(t-h)$ belongs to E, while (strictly speaking) $t \mapsto f(t+h)$ does not. Explain why the Laplace transform of $t \mapsto f(t+h)$ exists.

1. Using the substitution $\hat{t} = t - h$ and since $f(t) = 0$ for $t < 0$, we get

$$\mathcal{L}(f(t-h))(s) = \int_0^\infty e^{-st} f(t-h)\,\mathrm{d}t = \int_{-h}^\infty e^{-s(\hat{t}+h)} f(\hat{t})\,\mathrm{d}\hat{t} = e^{-hs}\int_0^\infty e^{-s\hat{t}} f(\hat{t})\,\mathrm{d}\hat{t} = e^{-hs}\mathcal{L}f(s).$$

2. Similarly, we use the substitution $\hat{t} = t + h$. But now, we have to add zero as $0 = \int_0^h e^{-st} f(t)\,\mathrm{d}t - \int_0^h e^{-st} f(t)\,\mathrm{d}t$ to get the desired form,

$$\begin{aligned}
\mathcal{L}(f(t+h))(s) &= \int_0^\infty e^{-st} f(t+h)\,\mathrm{d}t = \int_h^\infty e^{-s(\hat{t}-h)} f(\hat{t})\,\mathrm{d}\hat{t} = e^{hs}\int_h^\infty e^{-s\hat{t}} f(\hat{t})\,\mathrm{d}\hat{t} \\
&= e^{hs}\left(\int_h^\infty e^{-s\hat{t}} f(\hat{t})\,\mathrm{d}\hat{t} + \int_0^h e^{-st} f(t)\,\mathrm{d}t - \int_0^h e^{-st} f(t)\,\mathrm{d}t\right) \\
&= e^{hs}\left(\int_0^\infty e^{-s\hat{t}} f(\hat{t})\,\mathrm{d}\hat{t} - \int_0^h e^{-st} f(t)\,\mathrm{d}t\right) = e^{hs}\Big(\mathcal{L}f(s) - \int_0^h e^{-st} f(t)\,\mathrm{d}t\Big),
\end{aligned}$$

proving the second assertion. □

Example 3.11 Consider $f : \mathbb{R} \to \mathbb{R}$ given by the identity on $\mathbb{R}_{\geq 0}$ and by 0 on $\mathbb{R}_{<0}$, and let $h > 0$. Let us find the Laplace transform of $g_1 : t \mapsto f(t-h)$ and $g_2 : t \mapsto f(t+h)$. Firstly, we know from Example 3.4 that $\mathcal{L}f(s) = \frac{1}{s^2}$. Secondly, applying the first shifting property, we obtain

$$\mathcal{L}g_1(s) = e^{-hs}\mathcal{L}f(s) = \frac{e^{-hs}}{s^2}\,.$$

For the second function we use the second shifting property and a similar calculation as in Example 3.4 Part 2 and we get

$$\begin{aligned}\mathcal{L}g_2(s) &= e^{sh}\Big(\mathcal{L}f(s) - \int_0^h e^{-st} f(t)\,\mathrm{d}t\Big) = e^{sh}\left(\frac{1}{s^2} - \int_0^h e^{-st}\,t\,\mathrm{d}t\right)\\ &= e^{sh}\left(\frac{1}{s^2} + \frac{e^{-st}t}{s}\Big|_{t=0}^{h} + \int_0^h \frac{e^{-st}}{s}\,\mathrm{d}t\right) = e^{sh}\left(\frac{1}{s^2} + \frac{he^{-sh}}{s} + \frac{e^{-st}}{s^2}\Big|_{t=0}^{h}\right)\\ &= e^{sh}\left(\frac{1}{s^2} + \frac{he^{-sh}}{s} + \frac{e^{-sh}}{s^2} - \frac{1}{s^2}\right) = \frac{hs+1}{s^2}\,.\end{aligned}$$

Proposition 3.12 (Contraction) *Let $f \in E$ and $\lambda \in \mathbb{C}$. Then $t \mapsto e^{-\lambda t} f(t)$ belongs to E and*

$$\mathcal{L}(e^{-\lambda t} f(t))(s) = \mathcal{L}f(s+\lambda)\,.$$

Proof A few short considerations show that the function lies in E and we get

$$\mathcal{L}(e^{-\lambda t} f(t))(s) = \int_0^\infty e^{-st} e^{-\lambda t} f(t)\,\mathrm{d}t = \int_0^\infty e^{-(s+\lambda)t} f(t)\,\mathrm{d}t = \mathcal{L}f(s+\lambda).$$

□

Example 3.13 Find the Laplace transform of $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $f(t) = e^{-2t}t$. By Example 3.4, we know that $\mathcal{L}(t)(s) = \frac{1}{s^2}$, and hence, by Proposition 3.12, we have $\mathcal{L}f(s) = \frac{1}{(s+2)^2}$.

Exercise 3.14 *Test your understanding of the properties derived thus far by solving the following exercises. Find the Laplace transforms of*

1. $f_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *defined by* $f_1(t) = e^{-3t}\sin(t)$,
2. $f_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *defined by* $f_2(t) = t - 2$, *and*
3. $f_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *defined by* $f_3(t) = 4e^{-3t}\sin(t) + 2(t-2)$.

Lemma 3.15 (Derivative property) *Let $f \in E$ be a differentiable function, such that $f' \in E$. Then the Laplace transform of f' is given by*

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0)\,.$$

Proof Integrating by parts gives us

$$\mathcal{L}f'(s) = \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t = e^{-st} f(t)\Big|_{t=0}^{\infty} + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = -f(0) + s\mathcal{L}f(s)\,,$$

proving the assertion. □

Remark 3.16 This is an important result and lies behind future applications that involve solving linear differential equations. The key property is that the transform of a derivative f' does not itself involve a derivative.

Example 3.17 Let us determine the Laplace transform of the cosine function based on the given Laplace transform of the sine function. We know that $\mathcal{L}(\sin(at))(s) = \frac{a}{a^2+s^2}$. Since $\cos(at) = \frac{1}{a}(\sin(at))'$, we get

$$\mathcal{L}(\cos(at))(s) = \frac{1}{a}\mathcal{L}((\sin(at))')(s) = \frac{1}{a}\frac{sa}{a^2+s^2} = \frac{s}{a^2+s^2}\,.$$

Compare this result to the the result you got in Exercise 3.5.

We can actually generalise Lemma 3.15 to any n-th derivative, provided that the derivatives stay in E. Note that here f and its first $n-1$ derivatives appear, which are all evaluated at 0.

Proposition 3.18 (General derivative property) *Let f be $n \in \mathbb{N}$ times differentiable such that $f^{(n)} \in E$, then*

$$\mathcal{L}f^{(n)}(s) = s^n\mathcal{L}f(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)\,. \tag{3.2}$$

Proof We prove this result by induction. With Lemma 3.15 we have already established the induction start for $n = 1$. So for the induction hypothesis we assume that Equation (3.2) holds for $n-1$. We want to show that it also holds for n. Using integration by parts we get

$$\begin{aligned}
\mathcal{L}f^{(n)}(s) &= \int_0^\infty e^{-st} f^{(n)}(t)\,\mathrm{d}t = e^{-st} f^{(n-1)}(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f^{(n-1)}(t)\,\mathrm{d}t \\
&= -f^{(n-1)}(0) + s\mathcal{L}f^{(n-1)}(s) \overset{\text{IH}}{=} -f^{(n-1)}(0) + s\left(s^{n-1}\mathcal{L}f(s) - \sum_{k=1}^{n-1} s^{n-1-k} f^{(k-1)}(0)\right) \\
&= s^n\mathcal{L}f(s) - \sum_{k=1}^{n-1} s^{n-k} f^{(k-1)}(0) - f^{(n-1)}(0) = s^n\mathcal{L}f(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)\,.
\end{aligned}$$

Hence, we have shown Equation (3.2) and by induction it holds for all $n \in \mathbb{N}$, given that f is n times differentiable. □

Proposition 3.19 (Differentiability) *Let $\mathcal{L}f$ be the Laplace transform of a given function f. Then $\mathcal{L}f$ is infinitely differentiable and its derivatives are given for $n \in \mathbb{N}$ by*

$$(\mathcal{L}f)^{(n)} = \mathcal{L}((-t)^n f(t)). \tag{3.3}$$

Proof We prove the assertion by induction. First we show the induction start for $n = 1$.

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}f(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} e^{-st} f(t)\,\mathrm{d}t = \int_0^\infty -te^{-st} f(t)\,\mathrm{d}t = \mathcal{L}(-tf(t))(s)\,.$$

Next, assume that Equation (3.3) holds for $n-1$ and we show that it also holds for n. Basically the same calculation yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}(\mathcal{L}f)^{(n-1)}(s) &\overset{\text{IH}}{=} \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}((-t)^{n-1} f(t))(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}(-t)^{n-1} f(t)\,\mathrm{d}t \\
&= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} e^{-st}(t)^{n-1} f(t)\,\mathrm{d}t = \int_0^\infty e^{-st}(-t)^n f(t)\,\mathrm{d}t = \mathcal{L}((-t)^n f(t))(s)\,.
\end{aligned}$$

Thus, by induction, Equation (3.3) holds for any $n \in \mathbb{N}$, proving the assertion. □

Example 3.20 Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given by $f(t) = e^{-at}$. We want to find $(\mathcal{L}f)''$. So one approach would be to find $\mathcal{L}f$ and differentiate it twice. Ok, let's do that: set $\Theta = 1_{\mathbb{R}_{\geq 0}}$ and note that $f(t) = e^{-at}\Theta(t)$ (extended to $\mathbb{R}$). So with the contraction property from Proposition 3.12, we get $\mathcal{L}f(s) = \mathcal{L}\Theta(s+a)$, and from Example 3.4 we know $\mathcal{L}\Theta(s+a) = \frac{1}{s+a}$. So we are left to differentiate this expression twice, which yields

$$(\mathcal{L}f)''(s) = (\mathcal{L}\Theta)''(s+a) = \frac{2}{(s+a)^3}\,.$$

In another approach, we could make use of Proposition 3.19. Thereby, and by linearity, we have $(\mathcal{L}f)''(s) = \mathcal{L}((-t)^2 e^{-at})(s) = \mathcal{L}(t^2 e^{-at})(s)$. Now, using the contraction property again and Exercise 3.5, we get

$$(\mathcal{L}f)''(s) = \mathcal{L}(t^2 e^{-at})(s) = \mathcal{L}(t^2)(s+a) = \frac{2}{(s+a)^3}\,.$$

Summarising, the first approach includes basic functions, but we are left with differentiating, whereas the second approach includes more involved function, but we do not need to differentiate. So the choice really depends on your preferences and the underlying problem.

Proposition 3.21 (Integration property) *Let $f \in E$ and let $\mathcal{L}f$ be its Laplace transform. The Laplace transform of $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by $g(t) = \int_0^t f(u)\,\mathrm{d}u$ is given by*

$$\mathcal{L}g(s) = \frac{\mathcal{L}f(s)}{s} \tag{3.4}$$

Proof Note that $g'(t) = f(t)$ for all points t of continuity of f. Hence, by Lemma 3.15 we have

$$s\mathcal{L}g(s) = \mathcal{L}g'(s) + g(0) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t + 0 = \mathcal{L}f(s)\,.$$

Dividing both side by s yields the assertion. □

Exercise 3.22 *Thus far, we have seen different ways to calculate the Laplace transform of the same functions. In this exercise, you will calculate known Laplace transforms through yet another approach. You can check your results from the examples above.*

1. Using Proposition 3.21 and your knowledge of $\mathcal{L}(\sin(at))$ find the Laplace transform of the function $t \mapsto \cos(at)$.
2. Similarly, find the Laplace transform of $t \mapsto t^2$ given your knowledge of $\mathcal{L}(t)$.

State the domain of the functions you use and explain your choice.

Proposition 3.23 (Integration) *Let $f \in E$. The integral of its Laplace transform is given by*

$$\int_s^\infty \mathcal{L}f(u)\,du = \mathcal{L}(t^{-1}f(t))(s). \tag{3.5}$$

Proof Set $g(t) = t^{-1}f(t)$. By Proposition 3.19 we have $\mathcal{L}f = \mathcal{L}(tg(t)) = -(\mathcal{L}g)'$. Hence, for s, s_0 with sufficiently large real parts we have

$$\mathcal{L}g(s) - \mathcal{L}g(s_0) = \int_s^{s_0} \mathcal{L}f(u)\,\mathrm{d}u\,.$$

Since $\lim_{\Re(s_0)\to\infty} \mathcal{L}g(s_0) = 0$, the assertion follows. □

Example 3.24 Find the Laplace transform of the function given by $f(t) = t^2$, given that we know $\mathcal{L}(t^3)(s) = \frac{6}{s^4}$. Using Proposition 3.23 we have

$$\mathcal{L}(t^2)(s) = \mathcal{L}(t^{-1}t^3)(s) = \int_s^\infty \frac{6}{t^4}\,\mathrm{d}t = -\frac{2}{t^3}\Big|_{t=s}^{\infty} = \frac{2}{s^3}\,.$$

As you have already seen in Exercise 3.5 and Exercise 3.22.

Definition 3.25 (Convolution) The *convolution* of the functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ defined as

$$(f * g)(t) = f * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau))\,\mathrm{d}\tau\,. \tag{3.6}$$

If the functions f, g are only defined on $\mathbb{R}_{\geq 0}$, then we can truncate the integration limits and we define

$$f * g(t) = \int_0^t f(\tau)g(t-\tau))\,\mathrm{d}\tau\,. \tag{3.7}$$

Please do not write $f(t) * g(t)$. If you wondering why, then try to find the differences between $f(t) * g(t-t_0)$, $f(t-t_0) * g(t-t_0)$, and $f * g(t-t_0)$.

Example 3.26 Let us visualise what exactly the convolution does. Let f be defined by its green graph and let g be given by its dashed black graph in Figure 3.1. The solid black graph represents the function $t \mapsto g(t-\tau)$. In the top left subfigure, $\tau = -2.5$, indicated by the dash-dotted vertical line. Finally, the red graph represents $f * g$. Notice that as the graph of g wanders from the left to the right, it intersects with the graph of f, and the integral over their product becomes positive, indicated by the green filling.

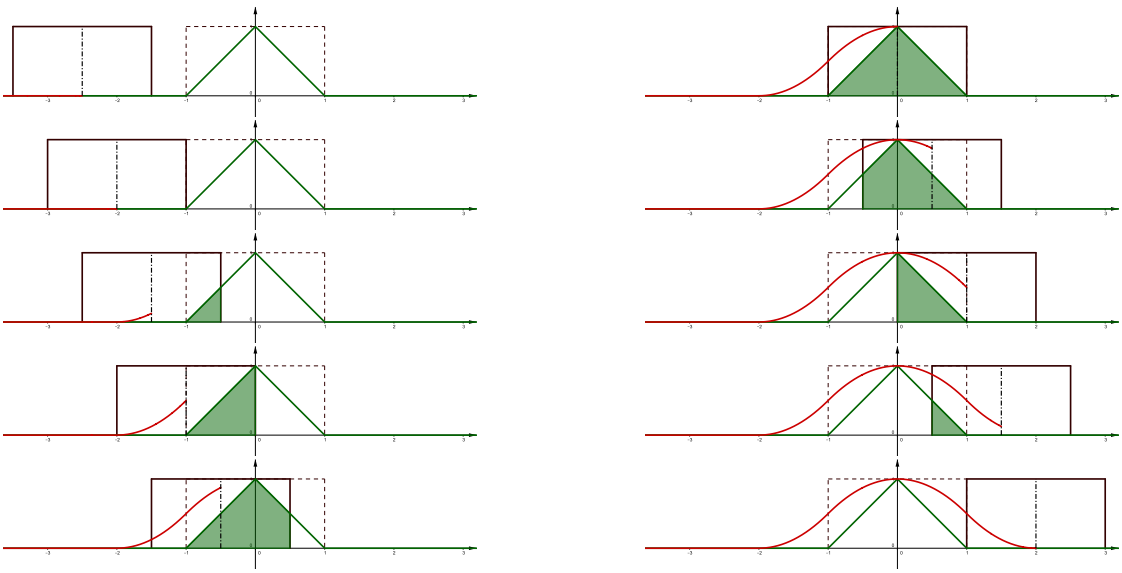


Figure 3.1: Visualisation of the convolution of f and g.

Exercise 3.27 *Analyse the green and black graphs in Figure 3.1 and define the functions f, g. Now calculate $f * g$, the function corresponding to the red graph.*

The convolution satisfies the following algebraic properties.

Exercise 3.28 *Given functions $f, g, h : \mathbb{R} \to \mathbb{R}$, show that the convolution satisfies*

1. *Commutativity: $f * g = g * f$,*
2. *Associativity: $(f * g) * h = f * (g * h)$,*
3. *Distributivity: $f * (g + h) = f * g + f * h$.*

Proposition 3.29 *Let $f, g \in E$. Then $f * g \in E$ and its Laplace transform is equal to the multiplication of the Laplace transforms of f and g,*

$$\mathcal{L}\{f * g\} = \mathcal{L}f\mathcal{L}g\,. \tag{3.8}$$

Proof First of all we show that $f * g \in E$. Since f and g are in E, there are constants $\alpha \in \mathbb{R}$ and $C > 0$ such that $|f(t)|, |g(t)| \le Ce^{\alpha t}$ for all $t > 0$. Hence,

$$|f * g(t)| \le \int_0^t |f(t-\tau)||g(\tau)|\,\mathrm{d}\tau \le C^2 \int_0^t e^{\alpha(t-\tau)}e^{\alpha\tau}\,\mathrm{d}\tau = C^2 t e^{\alpha t} \le \widehat{C}e^{\widehat{\alpha}t}\,,$$

for suitable $\widehat{\alpha} > \alpha$ and $\widehat{C} > 0$. The other two conditions are left as an exercise. We calculate the Laplace transform by interchanging the integrals,

$$\begin{aligned}\mathcal{L}\{f * g\}(s) &= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau\right)\mathrm{d}t = \int_0^\infty e^{-s\tau}f(\tau)\left(\int_\tau^\infty e^{-s(t-\tau)}g(t-\tau)\,\mathrm{d}t\right)\mathrm{d}\tau \\ &= \int_0^\infty e^{-s\tau}f(\tau)\,\mathcal{L}g(s)\,\mathrm{d}\tau = \mathcal{L}f(s)\mathcal{L}g(s)\,,\end{aligned}$$

proving the proposition. □

Convolutions occur in many different branches of engineering, particularly when signals are being processed. However, we will mainly be concerned with their application in the inverse Laplace transform.

3.3 The inverse Laplace transform

The inverse Laplace transform is a path integral in the complex plane,

$$f(t) = \mathcal{L}^{-1}F(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} e^{st}F(s)\,\mathrm{d}s\,,$$

for sufficiently large c (whatever that means). For our applications, however, we will not use this formula, known as *Mellin's inverse formula*, but we will concentrate on standard Laplace transforms and read them backwards.

The underlying result is known as *Lerch's theorem*, which states that the *Laplace operator is injective*. This means that if f and g are two different functions (different on a set of positive Lebesgue measure) then their Laplace transforms differ. In other words, if F has an inverse Laplace transform, then $f = \mathcal{L}^{-1}F$ is uniquely determined.

In particular, this means we can reverse the Doetsch-symbol and write $\mathcal{L}f \bullet\!\!-\!\!\circ\, f$, meaning the function $\mathcal{L}f$ has inverse Laplace transform f. For example, we can write

$$\frac{1}{s^2} \bullet\!\!-\!\!\circ\, t\,, \quad \text{since we already know} \quad t \circ\!\!-\!\!\bullet\, \frac{1}{s^2}\,.$$

So if we start with an $f \in E$ and then calculate $\mathcal{L}f$, we can come back with $\mathcal{L}^{-1}\mathcal{L}f = f$.

Example 3.30 Let $\mathcal{L}f(s) = \frac{1}{s(s^2+1)}$ be given. In this example, we want to find the function f. We already know that

$$\mathcal{L}(1)(s) = \frac{1}{s} \text{ and } \mathcal{L}(\sin(t))(s) = \frac{1}{s^2+1}, \text{ so we have } \mathcal{L}f(s) = \frac{1}{s}\frac{1}{s^2+1} = \mathcal{L}(1)(s)\cdot\mathcal{L}(\sin(t))(s).$$

Applying Proposition 3.29, we get

$$\mathcal{L}f(s) = \mathcal{L}(1 * \sin(t))(s).$$

Hence, we calculate

$$f(t) = \mathcal{L}^{-1}\mathcal{L}(1*\sin(t))(t) = 1*\sin(t) = \int_0^t 1\cdot\sin(u)\,\mathrm{d}u = -\cos(u)\Big|_0^t = 1-\cos(t).$$

Proposition 3.31 *Given the Laplace transform $\mathcal{L}f$ of f, then we have the following correspondence of limits,*

- $\lim_{t\downarrow 0} f(t) = \lim_{s\to\infty} s\mathcal{L}f(s)$,
- $\lim_{t\to\infty} f(t) = \lim_{s\downarrow 0}(s\mathcal{L}f(s))$, *if $\mathcal{L}f$ has no singularities in $\Re z \geq 0$ except at $s = 0$.*

Exercise 3.32 *Find a short proof of Proposition 3.31 under the additional assumption that f is differentiable and $f' \in E$.*
Hint: *Use the derivative property (Lemma 3.15) and the fact that $\lim_{\Re s\to\infty}\mathcal{L}f(s) = 0$ for all $f \in E$.*

Example 3.33 Let us find the limits of the function f as $t \to 0$ and $t \to \infty$ given its Laplace transform $\mathcal{L}f(s) = \frac{1}{s^2+4}$.
Using Proposition 3.31 we directly get

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\mathcal{L}f(s) = \lim_{s\to\infty}\frac{s}{s^2+4} = 0,$$
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\frac{s}{s^2+4} = 0.$$

Explain the last equality in the first expression for $s \in \mathbb{C}$.

3.4 Laplace transformation of periodic functions

For the next part we need an auxiliary result. We will learn about geometric series. The following lemma is very useful.

Lemma 3.34 (Geometric series) *The geometric series converges if $|q| < 1$ and takes the value*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

Proof First we consider for $N \in \mathbb{N}$ the sum $\sum_{n=0}^{N} q^n$ and multiply it by $1-q$ to get

$$(1-q)\sum_{n=0}^{N} q^n = \sum_{n=0}^{N} q^n - \sum_{n=1}^{N+1} q^n = 1 + \cancel{\sum_{n=1}^{N} q^n} - \cancel{\sum_{n=1}^{N} q^n} - q^{N+1} = 1 - q^{N+1}.$$

This type of sum is called a *telescopic sum*, since it is actually just an expansion of $1-q^{N+1}$ by zeros $(q-q)+(q^2-q^2)+\ldots+(q^{N-1}-q^{N-1})$. So, solving for the sum we get

$$\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}\,.$$

Since $|q|<1$ we have $\lim_{n\to\infty} q^n = 0$ and thus,

$$\sum_{n=0}^{\infty} q^n = \lim_{n\to\infty}\sum_{n=0}^{N} q^n = \lim_{n\to\infty}\frac{1-q^{N+1}}{1-q} = \frac{1}{1-q}\,,$$

proving the assertion. □

Exercise 3.35 *Let $|q|<1$ and calculate the value of $\sum_{n=0}^{\infty}(-q)^n$, using a similar procedure as above (you essentially have to change only one thing).*

Theorem 3.36 (Laplace transform of periodic functions) *Let $f:\mathbb{R}_{\geq 0}\to\mathbb{R}$ be periodic with period $P>0$. Then*

$$\mathcal{L}f(s) = \frac{1}{1-e^{-sP}}\int_0^P e^{-st}f(t)\,\mathrm{d}t\,. \tag{3.9}$$

Proof First of all we write down the definition of the Laplace transform. Then we split the integral into intervals of length P and use periodicity of f, more precisely, we use Remark 0.7 for $n\in\mathbb{N}$. We get

$$\begin{aligned}\mathcal{L}f(s) &= \int_0^\infty e^{-st}f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty}\int_{nP}^{(n+1)P} e^{-st}f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty}\int_0^P e^{-s(t+nP)}f(t+nP)\,\mathrm{d}t\\ &= \sum_{n=0}^{\infty}\int_0^P e^{-st}e^{-snP}f(t)\,\mathrm{d}t = \left(\sum_{n=0}^{\infty}e^{-snP}\right)\left(\int_0^P e^{-st}f(t)\,\mathrm{d}t\right).\end{aligned}$$

Note that $\sum_{n=0}^{\infty}e^{-snP} = \sum_{n=0}^{\infty}(e^{-sP})^n$ is a geometric series and $e^{-sP}<1$. Using Lemma 3.34 we get $\sum_{n=0}^{\infty}e^{-snP} = \frac{1}{1-e^{-sP}}$ and thus, the assertion follows. (Neat, huh?) □

Example 3.37 Consider the square wave from Example 0.16

$$f(x) = \begin{cases} -1 & \text{if } \lfloor 2x\rfloor \pmod 2 = 0\,,\\ 1 & \text{else.}\end{cases}$$

Since f is periodic with period 1 we get with help of Theorem 3.36

$$\begin{aligned}\mathcal{L}f(s) &= \frac{1}{1-e^{-s}}\int_0^1 e^{-st}f(t)\,\mathrm{d}t = \frac{1}{1-e^{-s}}\left(\int_0^{\frac12}e^{-st}\,\mathrm{d}t - \int_{\frac12}^1 e^{-st}\,\mathrm{d}t\right)\\ &= -\frac{1}{s}\frac{1}{(1-e^{-s})}\left(e^{-st}\Big|_0^{\frac12} - e^{-st}\Big|_{\frac12}^1\right) = -\frac{1}{s}\frac{1}{(1-e^{-s})}\left(e^{-\frac{s}{2}}-1-e^{-s}+e^{-\frac{s}{2}}\right)\\ &= \frac{1-2e^{-\frac{s}{2}}+e^{-s}}{s(1-e^{-s})}\,.\end{aligned}$$

Exercise 3.38 *We have now calculated the Laplace transform of the sine multiple times. Do it one more time using Theorem 3.36.*

3.5 Application of the Laplace transformation to ordinary differential equations

The Laplace transformation is used to solve homogeneous and non-homogeneous ordinary differential equations or systems of such equations. To understand the procedure, we consider a number of examples.

Suppose we are given a linear differential equation of the form

$$\begin{aligned} x'(t) + ax(t) &= \vartheta(t) \\ \text{subject to } x(0) &= x_0\,, \end{aligned}$$

where $a, x_0 \in \mathbb{R}$ and $\vartheta : \mathbb{R} \to \mathbb{C}$ so that the Laplace transform exists, that is $\vartheta \in E$. We saw that $\mathcal{L}x'(s) = s\mathcal{L}x(s) - x(0)$. So we can apply the Laplace operator to both sides of the differential equation, solve the equation, and then transform back. In more detail, we proceed as follows.

1. Apply the Laplace operator to both sides of the differential equation. Using linearity of the Laplace operator and then Lemma 3.15, we can write

$$\mathcal{L}\vartheta(s) = \mathcal{L}\{x' + ax\}(s) = \mathcal{L}\{x'\}(s) + a\mathcal{L}x(s) = s\mathcal{L}x(s) - x(0) + a\mathcal{L}x(s) = (s+a)\mathcal{L}x(s) - x_0\,.$$

2. So we can solve for $\mathcal{L}x(s)$ and get

$$\mathcal{L}x(s) = \frac{\mathcal{L}\vartheta(s) + x_0}{s+a}\,.$$

3. Finally, we can find the inverse Laplace transform $x(t) = \mathcal{L}^{-1}\{\mathcal{L}x\}(t)$, usually with help of Proposition 3.29.

Example 3.39 We are interested in the solution of the following ordinary differential equation

$$\begin{aligned} x'(t) + 2x(t) &= 2t - 4\,, \\ \text{subject to } x(0) &= 1\,. \end{aligned}$$

Following the steps from above, we get

1. Firstly, we apply the Laplace operator to the both parts of the equation and use its linearity. This together with Exercise 3.5 leads to

$$\mathcal{L}\{x' + 2x\} = \mathcal{L}(2t-4) \iff \mathcal{L}\{x'\} + 2\mathcal{L}x = 2\mathcal{L}(t) - 4\mathcal{L}(1) \iff s\mathcal{L}x(s) - x(0) + 2\mathcal{L}x(s) = \frac{2}{s^2} - \frac{4}{s}.$$

2. We solve for $\mathcal{L}x$ and obtain

$$(s+2)\mathcal{L}x(s) - 1 = \frac{2}{s^2} - \frac{4}{s} \iff \mathcal{L}x(s) = \frac{2}{s^2(s+2)} - \frac{4}{s(s+2)} + \frac{1}{s+2}\,.$$

3. Finally, we can use Exercise 3.5, Proposition 3.12 and Proposition 3.29 to find the inverse Laplace transform as

$$\begin{aligned} x(t) &= 2\mathcal{L}^{-1}\left(\frac{1}{s^2(s+2)}\right) - 4\mathcal{L}^{-1}\left(\frac{1}{s(s+2)}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \\ &= 2(t * e^{-2t})(t) - 4(1 * e^{-2t})(t) + e^{-2t} = 2\int_0^t (t-\tau)e^{-2\tau}\mathrm{d}\tau - 4\int_0^t 1\cdot e^{-2\tau}\mathrm{d}\tau + e^{-2t} \\ &= t + \frac{e^{-2t}-1}{2} + 2(e^{-2t}-1) + e^{-2t} = t + \frac{7e^{-2t}-5}{2}\,. \end{aligned}$$

We can even solve differential equations of higher order using a similar procedure. In this section, we derive the general result with help of general derivative property in Proposition 3.18.

Theorem 3.40 *For $n \in \mathbb{N}$, two sets of constants $\{a_k\}_{k=0}^n \subset \mathbb{R}$ and $\{c_k\}_{k=0}^{n-1} \subset \mathbb{R}$, and a function $\vartheta : \mathbb{R} \to \mathbb{C}$, $\vartheta \in E$ with Laplace transform $\mathcal{L}\vartheta$, a solution to the (non-)homogeneous linear differential equation of order n*

$$\sum_{k=0}^{n} a_k x^{(k)} = \vartheta(t),$$
$$\text{subject to } x^{(k)}(0) = c_k, \text{ for all } k \in \{0, \dots, n-1\}$$

is for $t \geq 0$ given by

$$x(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}\vartheta(s) + \sum_{k=1}^{n} \sum_{j=1}^{k} a_k s^{k-j} c_{j-1}}{\sum_{k=0}^{n} a_k s^k} \right)(t).$$

Proof In analogy to the procedure outlined above, we follow three steps.

Firstly, we apply the Laplace operator $\mathcal{L}$ to both sides of the differential equation, then use linearity and Proposition 3.18 to get

$$\begin{aligned} \mathcal{L}\{\vartheta\}(s) &= \mathcal{L}\left\{\textstyle\sum_{k=0}^{n} a_k x^{(k)}\right\}(s) = a_0 \mathcal{L}x(s) + \sum_{k=1}^{n} a_k \mathcal{L}\{x^{(k)}\}(s) \\ &= a_0 \mathcal{L}x(s) + \textstyle\sum_{k=1}^{n} a_k \left(s^k \mathcal{L}x(s) - \sum_{j=1}^{k} s^{k-j} x^{(j-1)}(0) \right) \\ &= \textstyle\sum_{k=0}^{n} a_k s^k \mathcal{L}x(s) - \sum_{k=1}^{n} \sum_{j=1}^{k} a_k s^{k-j} c_{j-1}. \end{aligned}$$

Secondly, we can solve this equation for $\mathcal{L}f$ and get

$$\mathcal{L}x(s) = \frac{\mathcal{L}\vartheta(s) + \sum_{k=1}^{n} \sum_{j=1}^{k} a_k s^{k-j} c_{j-1}}{\sum_{k=0}^{n} a_k s^k}. \tag{3.10}$$

Finally, the theorem follows by applying the inverse Laplace operator to both sides. □

Notice, if $x^{(k)}(0) = 0$, for all $k \in \{0, \dots, n-1\}$, then Equation (3.10) takes the form

$$\mathcal{L}x(s) = \frac{\mathcal{L}\vartheta(s)}{\sum_{k=0}^{n} a_k s^k}.$$

In Example 3.39, we needed to calculated the inverse Laplace transform of products like $\frac{1}{s}\frac{1}{(s+c)}$, just as Theorem 3.40 suggests. In this case, we made use of Proposition 3.29 and calculated the appropriate convolutions in step 3. In Example 3.42, we will see another trick that can be used.

For this, recall the *partial fraction decomposition (Partialbruchzerlegung)* of rational functions, that is, fractions such that their numerators and denominators are polynomials. It is somewhat the inverse procedure of finding a common denominator when adding multiple fractions. In general, we have to following result.

Lemma 3.41 (Partialbruchzerlegung über $\mathbb{R}$) *For any rational function $R : X \subset \mathbb{R} \to \mathbb{R}$ with m different real poles x_k of order p_k and n different (up to complex conjugation) complex poles z_k of order q_k can be uniquely represented as*

$$R(x) = P(x) + \sum_{k=1}^{m} \sum_{\ell=1}^{p_k} \frac{a_{k\ell}}{(x - x_k)^\ell} + \sum_{k=1}^{n} \sum_{\ell=1}^{q_k} \frac{b_{k\ell} x + c_{k\ell}}{(x - z_k)^\ell (x - \bar{z}_k)^\ell},$$

where P is a polynomial and all $a_{k\ell}, b_{k\ell}, c_{k\ell} \in \mathbb{R}$.

Consider the following example.

Example 3.42 We want to find a solution to the following differential equation,

$$x''(t) + 5x'(t) + 4x(t) = 10\,,$$
$$\text{subject to } x'(0) = x(0) = 0\,.$$

We use Theorem 3.40 and note that $x'(0) = x(0) = 0$, so that we get the form

$$\mathcal{L}x(s) = \frac{10\mathcal{L}(1)}{4+5s+s^2} = \frac{10}{s(4+5s+s^2)} = \frac{10}{s(s+1)(s+4)}\,. \tag{3.11}$$

We do not know the inverse Laplace transform of this form. But we notice that this is a rational function with three poles of order 1. So with Lemma 3.41 we know that it can be decomposed into a sum of fractions of the form,

$$\frac{10}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}\,, \tag{3.12}$$

where the constants A, B, and C have to be determined by coefficient comparison. Hence, multiplying both sides by $s(s+1)(s+4)$ we get

$$10 = A(s+1)(s+4) + Bs(s+4) + Cs(s+1) = (A+B+C)s^2 + (5A+4B+C)s + 4A\,,$$

which leads us to

$$A+B+C = 0, \quad 5A+4B+C = 0, \quad \text{and } 4A = 10.$$

From there we find $A = \frac{10}{4}, B = -\frac{10}{3}, C = \frac{10}{12}$, and therefore,

$$\mathcal{L}x(s) = 10\Big(\frac{1}{4s} - \frac{1}{3(s+1)} + \frac{1}{12(s+4)}\Big).$$

But this form is familiar, since we know that $1 \circ\!\!-\!\!\bullet\, s^{-1}$. Using the contraction property in Proposition 3.12 we get

$$x(t) = 10\Big(\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\Big).$$

So instead of working with the convolution to help us find the inverse Laplace transformation, we decomposed the fraction into a sum of 'simpler' fractions and used linearity of $\mathcal{L}^{-1}$.

Let us consider an example, where the rational function has only complex roots.

Example 3.43 Consider the following differential equation with initial conditions $x'(0) = x(0) = 0$,

$$2x''(t) + 4x(t) = \cos(3t)\,.$$

Applying Theorem 3.40 we find that

$$\mathcal{L}x(s) = \frac{\mathcal{L}(\cos(3t))(s)}{4+2s^2} = \frac{s}{s^2+9}\cdot\frac{1}{4+2s^2}\,.$$

Of course we know that $\mathcal{L}^{-1}(\frac{s}{s^2+9})(t) = \cos(3t)$ and we see that $\frac{1}{4+2s^2}$ has some correspondence with the sine (indeed, using the time scaling property in Proposition 3.8 we find $(2\sqrt{2})^{-1}\sin(\sqrt{2}t) \circ\!\!-\!\!\bullet\, (2s^2+4)^{-1}$). We have now two possibilities to continue. We could calculate the convolution of the two as Proposition 3.29 suggests. However, this seems tedious. So let us decompose this

fraction into a sum of fractions. We notice, that both s^2+9 and $4+2s^2$ have no real roots, since $s^2+9=(s-3i)(s+3i)$ and $2s^2+4=2(s-i\sqrt{2})(s+i\sqrt{2})$. So by Lemma 3.41, we know that it can be expressed as

$$\frac{s}{(s^2+9)(4+2s^2)}=\frac{A+Bs}{s^2+9}+\frac{C+Ds}{2s^2+4}.$$

Now we can proceed as in Example 3.42, and we will find $A=0$, $B=-\frac{1}{14}$, $C=0$, and $D=\frac{1}{7}$, so that

$$\frac{s}{(s^2+9)(4+2s^2)}=\frac{1}{14}\left(\frac{s}{s^2+2}-\frac{s}{s^2+9}\right).$$

But we know the inverse Laplace of the individual terms (with Exercise 3.5), so that by linearity, we find our solution

$$x(t)=\frac{1}{14}\Big(\cos(\sqrt{2}t)-\cos(3t)\Big),$$

and we are done! This procedure is usually faster than using convolutions. However, we have to remember which form the partial fraction decomposition takes, so that we actually can decompose it by just comparing coefficients.

A note to keep in mind: if we allow roots in $\mathbb{C}$, the numerators will always be constants. In this case, we would decompose as

$$\frac{s}{2(s^2+9)(2+s^2)}=\frac{1}{2}\left(\frac{A}{s-3i}+\frac{B}{s+3i}+\frac{C}{s-\sqrt{2}i}+\frac{D}{s+\sqrt{2}i}\right).$$

Comparing coefficients, we would get $A=B=-\frac{1}{14}$ and $C=D=\frac{1}{14}$. Plugging in and taking the inverse Laplace transform we would get

$$x(t)=\frac{1}{2}\frac{1}{14}\Big(e^{i\sqrt{2}t}+e^{-i\sqrt{2}t}-e^{i3t}-e^{i3t}\Big)=\frac{1}{14}\Big(\cos(\sqrt{2}t)-\cos(3t)\Big),$$

as already expected.

It is, in a sense, a trade-off. The coefficient comparison in the complex case may take longer by hand than the one in the real case. However, you can be certain that the numerators are constants and that you use the correct form. Additionally, it is arguably easier to remember the correspondence $e^{at}\circ\!\!-\!\!\bullet\,\frac{1}{s-a}$ than $\cos(at)\circ\!\!-\!\!\bullet\,\frac{s}{s^2+a^2}$ and $\sin(at)\circ\!\!-\!\!\bullet\,\frac{a}{s^2+a^2}$.

Let us conclude this section with two exercises.

Exercise 3.44 *Using the steps described above, find a solution to the following first-order linear differential equation,*

$$x'(t)+x(t)=e^t\,,\ \textit{subject to}\ \ x(0)=e^1=e\,.$$

You should get $x(t)=\frac{1}{2}(e^t-e^{-t}+2e^{1-t})$.

Exercise 3.45 *Argue that the 'Laplace-procedure' is not a great help in solving the following first-order* nonlinear *differential equation,*

$$x'(t)+(x(t))^2=0\,,\ \textit{subject to}\ \ x(0)=1\,.$$

Can you find a solution anyway? Maybe $x(t)=\frac{1}{1+t}$ *will do? But how to derive that, mh?*