Serie 12

1. Let $u(r,\theta)$ be a harmonic function in the disk

$$D = \{(r,\theta)|0 \leqslant r < R, -\pi < \theta \leqslant \pi\},$$

such that u is continuous in the closed disk $\overline{D}$ and satisfies

$$u(R,\theta) = \begin{cases} \sin^2 2\theta & |\theta| \leqslant \dfrac{\pi}{2}, \\ 0 & \dfrac{\pi}{2} < |\theta| \leqslant \pi. \end{cases}$$

   a) Evaluate $u(0,0)$ with out solving the PDE.
   b) Show that the inequality $0 < u(r,\theta) < 1$ holds at each point (r,θ) in the disk.

2. Find a function $u(r,\theta)$ harmonic in $\{2 < r < 4, 0 < \theta \leqslant 2\pi\}$, satisfying the boundary condition

$$u(2,\theta) = 0, \quad u(4,\theta) = \sin\theta.$$

3. Let $u(x,y)$ be the harmonic function in $D = \{(x,y)|x^2 + y^2 < 36\}$ which satisfies on ∂D the Dirichlet boundary condition

$$u(x,y) = \begin{cases} x & x \leqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

   a) Prove that $u(x,y) < \min\{x,0\}$ in D.

      Hint. *Prove that $u(x,y) < x$ and that $u(x,y) < 0$ in D. Notice that x is harmonic in D.*
   b) Evaluate $u(0,0)$ using the mean value principle.
   c) Using the separation of variables method, find the solution u in D.