Serie 13

1. Solve the following equation by the method of characteristics:
$$x^2 u_x - (y^2+1)u_y = ux^3, \quad u(1,y) = 2, x > 1$$

2. Let $D = \{x^2 + y^2 \leqslant 1\}$. Prove the solution of the following problem
$$\begin{aligned} \Delta u &= 0 \qquad (x,y) \in R^2 \backslash D \\ u(x,y) &= 1 \quad (x,y) \in \partial D \end{aligned}$$
is not unique.

   Remark. *This result does not contradict the Theorem 7.14 on p. 183 in [PR], since the domain $R^2 \backslash D$ is unbounded.*

3. Let $D = \{x^2 + y^2 \leqslant 1\}$ and $Q_T = [0, T] \times D$. Assume that $u : Q_T \to \mathbb{R}$ be a C^2 solution of the parabolic equation
$$u_t = k\Delta + yu_x - x^3 u_y.$$
   The parabolic boundary of Q_T is
$$\partial_P Q_T = \{0\} \times D \cup [0, T] \times \partial D$$
   Prove that u achieves its maximum on $\partial_P Q_T$.

   Hint. *Refer to the proof of Theorem 7.15, the maximum principle for the heat equation, in [PR].*

4. Find the weak solution of the following Cauchy problem with a discontinuous initial condition
$$\begin{aligned} u_t + u^2 u &= 0, \quad t \leqslant 0, x \in \mathbb{R}, \\ u(0,x) &= 3, \qquad x < 0, \\ u(0,x) &= 1, \qquad x > 0. \end{aligned}$$
   Hint. *Rewrite the equation in the form*
$$u_t + \partial_x F(u) = 0$$
   and use the Rankine-Hugoniot *condition, equation (2.62) on p. 47 in [PR].*

   Remark. *The Serie 4 is recommended for reviewing the topic conservation law.*

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).