Serie 4

The exercises with * are not required, students interested in them can try to solve.

1. Consider the equation

$$u_y + uu_x = 0$$

a) Show the following Cauchy problem has a differentiable solution for all positive time $y > 0$,

$$u(x,0) = \begin{cases} -1 & x \leqslant -\dfrac{\pi}{2}, \\ \sin x & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}, \\ 1 & x \geqslant \dfrac{\pi}{2}. \end{cases}$$

b) Show the solution of the following Cauchy problem becomes non-differentiable at some finite positive time and find the critical time y_c,

$$u(x,0) = \begin{cases} 1 & x \leqslant -\dfrac{\pi}{2}, \\ -\sin x & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}, \\ -1 & x \geqslant \dfrac{\pi}{2}. \end{cases}$$

c) Find a weak solution for the Cauchy problem with discontinuous initial condition

$$u(x,0) = \begin{cases} 1 & x < 1, \\ -2 & x > 1. \end{cases}$$

2. Consider the equation

$$u_y + u^2 u_x = 0$$

with the initial conditions

$$u(x,0) = \begin{cases} 1 & x \leqslant 0, \\ \sqrt{1 - \dfrac{x}{\alpha}} & 0 < x < \alpha, \\ 0 & x \geqslant \alpha. \end{cases}$$

a) Find the solution of the above Cauchy problem by the method of characteristics.

b)* Show that the solution will develop a singularity at some finite positive time. Find the critical time y_c.

c)* Let $\ell_0, \ell_{\frac{\alpha}{2}}, \ell_\alpha$ be the projections to (x, y)-plane of the characteristic curves through the points $(x, y, u) = (0, 0, 1), (\frac{\alpha}{2}, 0, \sqrt{\frac{1}{2}}), (\alpha, 0, 0)$. Show $\ell_0, \ell_{\frac{\alpha}{2}}, \ell_\alpha$ are straight lines and find their equations. Prove that they intersect at the point $(x, y) = (\alpha, y_c)$.

3. Consider the equation

$$u_y + u^2 u_x = 0$$

a) Formulate the weak problem by replacing the differential equation with an integral balance following the same idea for $u_u + uu_x = 0$ in [PR].

b)* Assume u be a weak solution that is a smooth function except for discontinuities along a curve $x = \gamma(y)$. Prove that

$$\gamma_y(y) = \frac{\frac{1}{3}(u_+^3 - u_-^3)}{u_+ - u_-} = \frac{1}{3}(u_+^2 + u_+u_- + u_-^2)$$

where

$$u_+(y) = \lim_{x \to \gamma(y)_+} u(x, y), \qquad u_-(y) = \lim_{x \to \gamma(y)_-} u(x, y).$$

c)* Apply the results to **Ex 2** to find a weak solution with a single discontinuity after the critical time y_c.

4. Consider the equation

$$u_{xx} + 2u_{xy} + [1 - q(y)]u_{yy} = 0,$$

where

$$q(y) = \begin{cases} -1 & y < -1, \\ 0 & |y| \leqslant 1, \\ 1 & y > 1. \end{cases}$$

a) Find the domain where the equation is hyperbolic, parabolic and elliptic.

b) For each of the above three domains, find the corresponding canonical transformation and the canonical form.
Hint: Consider linear transformations.

c) Draw the characteristics for the hyperbolic case.

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).