Serie 5

1. Consider the Cauchy problem

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty,\ t > 0.\\
&u(x,0) = \sin x, \quad -\infty < x < \infty,\\
&u_t(x,0) = 0, \quad -\infty < x < \infty.
\end{aligned}
$$

a) Solve the Cauchy problem and find the forward wave and backward wave.

b) Prove that $u(x, t + \frac{2\pi}{c}) = u(x,t)$.

2. A pressure wave generated as a result of an explosion satisfies the equation

$$P_{tt} - 16P_{xx} = 0$$

in the domain $\{(x,t) | -\infty < x < \infty, t > 0\}$, where $P(x,t)$ is the pressure at the point x and the time t. The initial conditions at the explosion time $t = 0$ are

$$
P(x,0) = \begin{cases} 10 & |x| \leqslant 1, \\ 0 & |x| > 1, \end{cases}
$$

$$
P_t(x,0) = \begin{cases} 1 & |x| \leqslant 1, \\ 0 & |x| > 1. \end{cases}
$$

A building is located at the point $x_0 = 10$. The engineer who designed the building determined that it will sustain a pressure up to $P = 6$. Find time t_0 when the pressure at the building is maximal. Will the building collapse?

3. $u(x,t)$ is the solution of the Cauchy problem

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty,\ t > 0.\\
&u(x,0) = f(x), \quad -\infty < x < \infty,\\
&u_t(x,0) = g(x), \quad -\infty < x < \infty.
\end{aligned}
$$

Let $\tilde{u}(x,t) = u(x, T - t)$. Prove that $\tilde{u}(x,t)$ is the solution of the Cauchy problem

$$
\begin{aligned}
&\tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0, \quad -\infty < x < \infty,\ t < T.\\
&\tilde{u}(x,T) = f(x), \quad -\infty < x < \infty,\\
&\tilde{u}_t(x,T) = -g(x), \quad -\infty < x < \infty.
\end{aligned}
$$

Remark. *The wave equation is time reversible, i.e. the equation is invariant under time reversing. It is not the case for the heat equation* $u_t - u_{xx} = 0$.

It is useful to know the equation being invariant under certain coordinate transformations. This exercise is a very simple example. From this exercise, one can easily conclude that for wave equation the property, which is true forward in time, will also be true backward in time.

Actually there is a set of transformations, under which the wave equation is invariant. It is called Lorentz transformation. The formula for Lorentz transformation is as follows:

$$(x,t) \mapsto \left(\frac{x - vt}{\sqrt{1 - (\frac{v}{c})^2}}, \frac{t - \frac{v}{c^2}x}{\sqrt{1 - (\frac{v}{c})^2}} \right)$$

4. Use the graphical method to solve the Cauchy problem

$$\begin{aligned} &u_{tt} - u_{xx} = 0, \quad -\infty < x < \infty,\, t > 0 \\ &u(x,0) = \begin{cases} 1 & x \in [-1,0] \cup [1,2], \\ 0 & x \in (-\infty,-1) \cup (0,1) \cap (2,\infty). \end{cases} \\ &u_t(x,0) = 0 \quad -\infty < x < \infty. \end{aligned}$$