Serie 6

1. a) Solve the following initial boundary value problem for a vibrating semi-infinite string which is fixed at $x = 0$:

$$\begin{aligned}
&u_{tt} - u_{xx} = 0 && 0 < x < \infty, t > 0 \\
&u(0,t) = 0 && t > 0 \\
&u(x,0) = f(x) && 0 \leqslant x < \infty, \\
&u_t(x,0) = g(x) && 0 \leqslant x < \infty,
\end{aligned}$$

where $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$ satisfy the compatibility conditions $f(0) = f''(0) = g(0) = 0$.

Hint. *Extend the functions f and g as odd functions $\tilde{f}$ and $\tilde{g}$ over the real line. Solve the Cauchy problem with initial data $\tilde{f}$ and $\tilde{g}$, and show that the restirction of this solution to the half-plane $x \geqslant 0$ is a solution of the problem. Recall that the soluiton of the Cauchy problem with odd data is odd. In particular, the solution with odd data is zero for $x = 0$ and all $t \geqslant 0$.*

b) Solve the problem with $f(x) = x^6$ and $g(x) = \sin(x)$, and evaluate $u(1,2)$. Is the solution classical?

2. Solve the problem

$$\begin{aligned}
&u_{tt} - u_{xx} = xt && -\infty < x < \infty, t > 0 \\
&u(x,0) = 0 && -\infty < x < \infty, \\
&u_t(x,0) = e^x && -\infty < x < \infty.
\end{aligned}$$

3. Solve the problem

$$\begin{aligned}
&u_{tt} - u_{xx} = 1 && -\infty < x < \infty, t > 0 \\
&u(x,0) = x^2 && -\infty < x < \infty, \\
&u_t(x,0) = 1 && -\infty < x < \infty.
\end{aligned}$$

4. Solve the problem

$$\begin{aligned}
&u_t - u_{xx} = 0 && 0 < x < \pi, t > 0 \\
&u(0,t) = u(\pi,t) = 0 && t \geqslant 0, \\
&u(x,0) = \sin(2x) + \frac{1}{2}\sin(3x) + 5\sin(5x) && 0 \leqslant x \leqslant \pi.
\end{aligned}$$