Serie 7

1. (Wave equation with Dirichlet boundary conditions)

a) Find a formal solution of the problem

$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < \pi, \quad t > 0, \\ &u(0,t) = u(\pi,t) = 0 && t \geqslant 0, \\ &u(x,0) = \sin^3 x && 0 \leqslant x \leqslant \pi, \\ &u_t(x,0) = \sin 2x && 0 \leqslant x \leqslant \pi. \end{aligned}$$

Hint. *Similar problem to example 5.2 in [PR], but notice that the boundary conditions here are Dirichlet boundary conditions. Use the trigonometric identities to obtain the Fourier expansion of* $\sin^3 x$*.*

b) Show that the above solution is classical.

2. (Wave equation with Newmann boundary conditions)
Find the solution of the problem:

$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = u_x(1,t) = 0 && t \geqslant 0, \\ &u(x,0) = \sin^2 \pi x && 0 \leqslant x \leqslant 1, \\ &u_t(x,0) = \cos \pi x && 0 \leqslant x \leqslant 1. \end{aligned}$$

Hint. *Similar problem to example 5.2 in [PR].*

3. (Heat equation with Newmann boundary conditions)
Solve the following problem

$$\begin{aligned} &u_t - 12u_{xx} = 0 && 0 < x < \pi, \quad t > 0, \\ &u_x(0,t) = u_x(\pi,t) = 0 && t \geqslant 0, \\ &u(x,0) = 1 + \sin^2 x && 0 \leqslant x \leqslant \pi. \end{aligned}$$

4. (Heat equation with mixed boundary condtions)

a) Find the solutions of the eigenvalue problem

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \lambda X = 0 \qquad 0 < x < \pi,$$
$$X(0) = \frac{\mathrm{d}X}{\mathrm{d}x}(\pi) = 0.$$

b) Solve the problem

$$\begin{aligned} &u_t - u_{xx} = 0 && 0 < x < \pi, \quad t > 0, \\ &u(0,t) = u_x(\pi,t) = 0 && t \geqslant 0, \\ &u(x,0) = \sin\frac{3x}{2} + \sin\frac{9x}{2} && 0 \leqslant x \leqslant \pi. \end{aligned}$$

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).