Solution 1

1. See the section 1.2 in [PR].
   - **(a)** 3. Ordnung; nicht linear, quasilinear.
   - **(b)** 1. Ordnung; nicht linear.
   - **(c)** 2. Ordnung; linear.
   - **(d)** 2. Ordnung; linear.
   - **(e)** 2. Ordnung; linear.
2. **(a)**

$$u_x = \alpha e^{\alpha x+\beta y}, u_y = \beta e^{\alpha x+\beta y},$$

we substitute u, u_x, u_y into the equation to get

$$(\alpha + 3\beta + 1)e^{\alpha x+\beta y} = 0,$$

so for any constants α, β satisfying the relation $\alpha + 3\beta + 1 = 0$, $u(x,y) = e^{\alpha x+\beta y}$ solves the equation.

(b)

$$u_{xx} = \alpha^2 e^{\alpha x+\beta y}, u_{yy} = \beta^2 e^{\alpha x+\beta y},$$

substituting into the equation, we get

$$(\alpha^2 + \beta^2)e^{\alpha x+\beta y} = 5e^{x-2y}.$$

Hence $\alpha = 1, \beta = -2$, $u(x,y) = e^{x-2y}$ is the only solution of the form $u(x,y) = e^{\alpha x+\beta y}$.

(c)

$$u_{xxxx} = \alpha^4 e^{\alpha x+\beta y}, u_{yyyy} = \beta^4 e^{\alpha x+\beta y}, u_{xxyy} = \alpha^2\beta^2 e^{\alpha x+\beta y},$$

substituting into the equation, we get

$$(\alpha^4 + \beta^4 + 2\alpha^2\beta^2)e^{\alpha x+\beta y} = 0.$$

Hence $\alpha = \beta = 0$, $u(x,y) = e^0 = 1$ is the only solution of the form $u(x,y) = e^{\alpha x+\beta y}$.

3. **(a)** Integrate $u_y = 2x$ for y,
$$u = 2xy + c(x).$$
Substitute into $u_x = 2(x + y)$,
$$2y + c'(x) = 2(x + y),$$
so $c'(x) = 2x$. Integrate for x,
$$c(x) = x^2 + c_1, \quad c_1 \in \mathbb{R}.$$
$u = 2xy + x^2 + c_1$ solves the equation. $u(0,0) = 0$ implies $c_1 = 0$.

(b) Any solution of the system is smooth, since $2(x + y)$ and Ax are smooth functions. So any solution satisfies $u_{xy} = u_{yx}$, contradicting
$$u_{xy} = \frac{\partial}{\partial y}(2(x + y)) = 2 \neq A = \frac{\partial}{\partial x}(Ax) = u_{yx}.$$

4. A particle starting its life at a boundary point dies at once. Thus
$$u(x, y, z) = 0, \quad (x, y, z) \in \partial D.$$
Consider an internal point (x, y, z), by the similar derivation in the book, we get the difference equation
$$u(x,y,z) = \delta t + \frac{1}{6}[u(x-\delta h, y, z) + u(x+\delta h, y, z) + u(x, y-\delta h, z) + u(x, y+\delta h, z) + u(x, y, z-\delta h) + u(x, y, z+\delta h)]$$
We expand all functions on the right hand side into a Taylor series, assuming $u \in C^4$,
$$u(x - \delta h, y, z) = u(x, y, z) - u_x(x, y, z) \cdot \delta h + \frac{1}{2}u_{xx}(x, y, z)(\delta h)^2 + O((\delta h)^3),$$
$$u(x + \delta h, y, z) = u(x, y, z) + u_x(x, y, z) \cdot \delta h + \frac{1}{2}u_{xx}(x, y, z)(\delta h)^2 + O((\delta h)^3),$$
etc.

Dividing by δt and taking the limit, we obtain
$$\Delta u = -\frac{1}{k}, \quad (x, y, z) \in D.$$

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).