Solution 11

1. From equation (7.9) on page 176 of [PR], let $D = [0,\pi] \times [0,\pi], \Gamma = \partial D$,

$$\begin{aligned}
0 = & \int_\Gamma \partial_n u \\
= & \int_0^\pi u_y(x,\pi)\mathrm{d}x + \int_0^\pi u_x(\pi,y)\mathrm{d}y \\
& - \int_0^\pi u_y(x,0)\mathrm{d}x - \int_0^\pi u_x(0,y)\mathrm{d}y \\
= & \int_0^\pi (x^2 - a)\mathrm{d}x \\
= & \frac{x^3}{3}|_0^\pi - a\pi \\
= & \frac{\pi^3}{3} - a\pi \\
\Rightarrow \quad & a = \frac{\pi^2}{3}.
\end{aligned}$$

2. a) Substitute ω and we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$
$$X'(0)Y(y) = X'(2\pi)Y(y) = 0$$

which implies the following eigenvalue problem

$$\begin{aligned}
X''(x) + \lambda X(x) = 0 \quad 0 < x < 2\pi, \\
Y''(y) - \lambda Y(y) = 0 \quad -1 < y < 1
\end{aligned}$$

From the boundary condition, we get

$$X'(0) = X'(2\pi) = 0,$$

hence the solutions for the eigenvalue problem of s are

$$\lambda_n = (\frac{n}{2})^2, \quad n = 0, 1, 2, 3...$$

and the corresponding eigenfunctions are

$$X_n(x) = \cos\frac{nx}{2}.$$

Then we can solve the equation for Y, the general solution is (see equation (7.43) in [PR])

$$\begin{aligned} Y_0(y) &= \alpha_0 y + \beta_0, \quad n = 0, \\ Y_n(y) &= \alpha_n \sinh\frac{n(y+1)}{2} + \beta_n \sinh\frac{n(y-1)}{2}, \quad n = 1, 2, 3... \end{aligned}$$

so

$$\begin{aligned} w_0(x, y) &= \alpha_0 y + \beta_0 \\ \omega_n(x, y) &= \cos\frac{nx}{2}\left(\alpha_n \sinh\frac{n(y+1)}{2} + \beta_n \sinh\frac{n(y-1)}{2}\right), \ n = 1, 2, 3... \end{aligned}$$

b) By the superposition principle, we can write u in the expansion of w_n

$$u(x, y) = \alpha_0 y + \beta_0 + \sum_{n=1}^{\infty} \cos\frac{nx}{2}\left(\alpha_n \sinh\frac{n(y+1)}{2} + \beta_n \sinh\frac{n(y-1)}{2}\right)$$

The boundary conditions implies

$$\begin{aligned} u(x, -1) &= -\alpha_0 + \beta_0 + \sum_{n=1}^{\infty} \beta_n \cos\frac{nx}{2} \sinh(-n) = 0 \\ u(x, 1) &= \alpha_0 + \beta_0 + \sum_{n=1}^{\infty} \alpha_n \cos\frac{nx}{2} \sinh n = 1 + \cos 2x \end{aligned}$$

hence we get the values of α_n, β_n

$$\begin{aligned} \alpha_0 &= \beta_0 = \frac{1}{2}, \\ \alpha_4 &= \frac{1}{\sinh 4}, \quad \alpha_n = 0, \ n \neq 0, 4, \\ \beta_n &= 0, \ n \neq 0. \end{aligned}$$

So the solution is

$$u(x, y) = \frac{y}{2} + \frac{1}{2} + \frac{1}{\sinh 4} \cos 2x \sinh 2(y+1)$$

3. For v_1, we have the Newmann boundary condition

$$
\begin{aligned}
(v_1)_x(0,y) &= a(x^2-y^2)_x = 2ax = 0, \\
(v_1)_x(\pi,y) &= \sin y + a(x^2-y^2)_x = \sin y + 2\pi a, \quad y \in [0,\pi], \\
(v_1)_y(x,0) &= 0,\ (v_1)_y(x,\pi) = 0, \qquad\qquad x \in [0,\pi]
\end{aligned}
$$

In order to guarantee the Newmann problem is solvable for v_1, the following must hold

$$
\begin{aligned}
0 = & \int_0^\pi (v_1)_y(x,\pi)\mathrm{d}x + \int_0^\pi (v_1)_x(\pi,y)\mathrm{d}y \\
& - \int_0^\pi (v_1)_y(x,0)\mathrm{d}x - \int_0^\pi (v_1)_x(0,y)\mathrm{d}y \\
= & \int_0^\pi (\sin y + 2\pi a)\mathrm{d}y \\
= & -\cos y|_0^\pi + 2\pi^2 a \\
= & \ 2 + 2\pi^2 a \\
\Rightarrow \quad & a = -\frac{1}{\pi^2}
\end{aligned}
$$

For v_2, we have the Newmann boundary condition

$$
\begin{aligned}
(v_2)_x(0,y) &= 0,\ (v_2)_x(\pi,y) = 0, \qquad\qquad y \in [0,\pi], \\
(v_2)_y(x,0) &= a(x^2-y^2)_y = 0, \\
(v_2)_y(x,\pi) &= -\sin x + a(x^2-y^2)_y = -\sin x - 2\pi a, \quad x \in [0,\pi].
\end{aligned}
$$

To guarantee the Newmann problem is solvable for v_1, the following must hold

$$
\begin{aligned}
0 = & \int_0^\pi (v_2)_y(x,\pi)\mathrm{d}x + \int_0^\pi (v_2)_x(\pi,y)\mathrm{d}y \\
& - \int_0^\pi (v_2)_y(x,0)\mathrm{d}x - \int_0^\pi (v_2)_x(0,y)\mathrm{d}y \\
= & \int_0^\pi (-\sin x - 2\pi a)\mathrm{d}x \\
= & \ \cos y|_0^\pi - 2\pi^2 a \\
= & -2 - 2\pi^2 a \\
\Rightarrow \quad & a = -\frac{1}{\pi^2}
\end{aligned}
$$

Hence $a = -\frac{1}{\pi^2}$.

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).