Serie 12

1. a) Using the mean value principle

$$\begin{aligned}
u(0,0) = &\ \frac{1}{2\pi}\int_{-\pi}^{\pi} u(R,\theta)\mathrm{d}\theta \\
= &\ \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\theta\,\mathrm{d}\theta \\
= &\ \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{\cos 4\theta}{2}\right)\mathrm{d}\theta \\
= &\ \frac{1}{2\pi}\left[\frac{1}{2}\cdot\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right) + \frac{1}{8}\sin 4\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] \\
= &\ \frac{1}{2\pi}\left[\frac{\pi}{2} + \left(\frac{1}{8}\sin(2\pi) - \frac{1}{8}\sin(-2\pi)\right)\right] \\
= &\ \frac{1}{4}.
\end{aligned}$$

b) By the weak maximum(minimum) principle (Theorem 7.5 in [PR]),

$$\min_{\partial D} u \leqslant u(r,\theta) \leqslant \max_{\partial D} u, \quad (r,\theta) \in D.$$

Clearly $\min_{\partial D} u = 0$ and $\max_{\partial D} u = \max\{\sin^2 2\theta, |\theta| \leqslant \frac{\pi}{2}\} = 1$, so

$$0 \leqslant u(r,\theta) \leqslant 1, \quad (r,\theta) \in D.$$

Since u is not constant, by the strong maximum principle (Theorem 7.10 and Remark 7.11 in [PR]), u cannot obtain its maximum or minimum in D, hence

$$0 < u(r,\theta) < 1, \quad (r,\theta) \in D.$$

2. Notice that $\sin\theta$ is an eigenfunction for the following eigenvalue problem

$$\begin{aligned}
&\Theta''(\theta) + \lambda\Theta(\theta) = 0, \\
&\Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi).
\end{aligned}$$

Recall that in section 7.7.2 in [PR], the above equation is derived by considering solutions of special form $R(r)\Theta(\theta)$. This reminds to consider the function of the form $w(r,\theta) = R(r)\sin\theta$. In polar coordinates,

$$\begin{aligned}\Delta w = &\quad w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta}\\ = &\quad \sin\theta\left(R_{rr}(r) + \frac{1}{r}R(r) - \frac{1}{r^2}R(r)\right)\end{aligned}$$

Hence $w(r,\theta) = R(r)\sin\theta$ is harmonic in $\{2 < r < 4, 0 \leqslant \theta \leqslant 2\pi\}$ for any function $R(r)$ satisfying the equation

$$r^2R_{rr}(r) + rR_r(r) - R(r) = 0, \quad 2 < r < 4,$$

The general solution for the above equation is given in section 7.7.2 p. 196 in [PR],

$$R(r) = Cr + Dr^{-1}$$

From the boundary condition

$$u(2,\theta) = 0, \quad u(4,\theta) = \sin\theta,$$

we need to solve C and D from

$$\begin{aligned}R(2) &= 2C + \frac{D}{2} = 0,\\ R(4) &= 4C + \frac{D}{4} = 1\end{aligned}$$

The solutions are $C = \frac{1}{3}, D = -\frac{4}{3}$. Hence

$$u(r,\theta) = \frac{r\sin\theta}{3} - \frac{4\sin\theta}{3r}$$

is the solution for the problem.

3. a) **$u < 0$:** By the strong maximum principle, since u is not constant

$$u(x,y) < \max_{\partial D} u(x,y) = 0$$

$u < x$: Consider the function $w(x,y) = u(x,y) - x$, since u and x are both harmonic, w is also harmonic. On ∂D

$$w(x,y) = u(x,y) - x = \begin{cases} 0 & x \leqslant 0 \\ -x & x > 0. \end{cases}$$

Since w is not constant, by the strong maximum principle

$$w(x,y) < \max_{\partial D} w(x,y) = 0$$

which implies

$$u(x,y) < x.$$

$u < 0$ and $u < x$ imply

$$u < \min\{x, 0\}.$$

b) In polar coordinates, $D = \{(r,\theta) | 0 < r < 6, 0 < \theta \leqslant 2\pi\}$,

$$u(r,\theta) = \begin{cases} 6\cos\theta & \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The mean value principle implies

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi}\int_0^{2\pi} u(6,\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos\theta\mathrm{d}\theta \\ &= \frac{1}{2\pi}(6\sin\theta)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= \frac{3}{\pi}(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}) \\ &= \frac{3}{\pi}(-1-1) \\ &= -\frac{6}{\pi} \end{aligned}$$

c) Apply the results in section 7.7.2 in [PR], the solution has the form

$$u(r,\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n(\alpha_n \cos n\theta + \beta_n \sin n\theta)$$

where

$$\begin{aligned} \alpha_0 &= \frac{1}{\pi}\int_0^{2\pi} u(6,\theta)\mathrm{d}\theta = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos\theta\,\mathrm{d}\theta = 2u(0,0) = -\frac{12}{\pi} \\ \alpha_n &= \frac{1}{\pi 6^n}\int_0^{2\pi} u(6,\theta)\cos n\theta\mathrm{d}\theta = \frac{1}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos\theta\cos n\theta\,\mathrm{d}\theta, \\ \beta_n &= \frac{1}{\pi 6^n}\int_0^{2\pi} u(6,\theta)\sin n\theta\mathrm{d}\theta = \frac{1}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos\theta\sin n\theta\,\mathrm{d}\theta, \end{aligned}$$

α_n**:** By the trigonometric formulas (7) in Appendix A.1 (p. 361) in [PR]

$$\cos\theta\cos n\theta = \frac{1}{2}[\cos(n+1)\theta + \cos(n-1)\theta]$$

hence

$$\begin{aligned} \alpha_n &= \frac{1}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos\theta\cos n\theta\,\mathrm{d}\theta \\ &= \frac{1}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3[\cos(n+1)\theta + \cos(n-1)\theta]\,\mathrm{d}\theta \end{aligned}$$

For $n = 1$

$$\begin{aligned} \alpha_1 &= \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1+\cos 2\theta)\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\left[\pi + \frac{1}{2}\sin 2\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right] \\ &= \frac{1}{2} + \frac{1}{4\pi}(\sin 3\pi - \sin\pi) \\ &= \frac{1}{2} \end{aligned}$$

For $n \neq 1$

$$\begin{aligned} \alpha_n &= \frac{3}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [\cos(n+1)\theta + \cos(n-1)\theta]\,\mathrm{d}\theta \\ &= \frac{3}{\pi 6^n}\left[\frac{1}{n+1}\sin(n+1)\theta + \frac{1}{n-1}\sin(n-1)\theta\right]\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= \frac{3}{\pi 6^n}\left\{\frac{1}{n+1}[\sin\frac{3(n+1)\pi}{2} - \sin\frac{(n+1)\pi}{2}] + \frac{1}{n-1}[\sin\frac{3(n-1)\pi}{2} - \sin\frac{(n-1)\pi}{2}]\right\} \\ &= \frac{3}{\pi 6^n}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\left[\sin\frac{3(n+1)\pi}{2} - \sin\frac{(n+1)\pi}{2}\right] \\ &= -\frac{6}{\pi 6^n}\frac{1}{n^2-1}\left[\sin\frac{3(n+1)\pi}{2} - \sin\frac{(n+1)\pi}{2}\right] \\ &= -\frac{6}{\pi 6^n}\frac{1}{n^2-1}\left[\sin\left(\frac{(n+1)\pi}{2} + (n+1)\pi\right) - \sin\frac{(n+1)\pi}{2}\right] \\ &= -\frac{6}{\pi 6^n}\frac{1}{n^2-1}\left((-1)^{n+1}-1\right)\sin\frac{(n+1)\pi}{2} \\ &= \begin{cases} -\dfrac{6}{\pi 6^{2k}}\dfrac{1}{(2k)^2-1}\left((-1)^{2k+1}-1\right)\sin\dfrac{(2k+1)\pi}{2} = \dfrac{(-1)^k 12}{6^{2k}[(2k)^2-1]\pi} & n = 2k \\ -\dfrac{6}{\pi 6^{2k+1}}\dfrac{1}{(2k+1)^2-1}\left((-1)^{2k+2}-1\right)\sin\dfrac{(2k+2)\pi}{2} = 0 & n = 2k+1 \end{cases} \end{aligned}$$

β_n**:** By the trigonometric formulas (9) in Appendix A.1 (p. 361) in [PR]

$$\cos\theta\sin n\theta = \frac{1}{2}[\sin(n+1)\theta + \sin(n-1)\theta]$$

hence

$$\begin{aligned}
\beta_n = &\quad \frac{1}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 6\cos\theta\sin n\theta\,\mathrm{d}\theta \\
= &\quad \frac{1}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 3[\sin(n+1)\theta + \sin(n-1)\theta]\,\mathrm{d}\theta
\end{aligned}$$

For $n = 1$

$$\begin{aligned}
\beta_1 = &\quad \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin 2\theta\,\mathrm{d}\theta \\
= &\quad -\frac{1}{2\pi}\frac{1}{2}\cos 2\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
= &\quad -\frac{1}{4\pi}(\cos 3\pi - \cos\pi) \\
= &\quad 0
\end{aligned}$$

For $n \neq 1$

$$\begin{aligned}
\beta_n = &\quad \frac{3}{\pi 6^n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [\sin(n+1)\theta + \sin(n-1)\theta]\,\mathrm{d}\theta \\
= &\quad \frac{3}{\pi 6^n}\left[-\frac{1}{n+1}\cos(n+1)\theta - \frac{1}{n-1}\cos(n-1)\theta\right]\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
= &\quad \frac{3}{\pi 6^n}\left\{-\frac{1}{n+1}[\cos\frac{3(n+1)\pi}{2} - \cos\frac{(n+1)\pi}{2}] - \frac{1}{n-1}[\cos\frac{3(n-1)\pi}{2} - \cos\frac{(n-1)\pi}{2}]\right\} \\
= &\quad \frac{3}{\pi 6^n}\left(-\frac{1}{n+1} + \frac{1}{n-1}\right)[\cos\frac{3(n+1)\pi}{2} - \cos\frac{(n+1)\pi}{2}] \\
= &\quad \frac{6}{\pi 6^n}\frac{1}{n^2-1}[\cos\frac{3(n+1)\pi}{2} - \cos\frac{(n+1)\pi}{2}] \\
= &\quad \frac{6}{\pi 6^n}\frac{1}{n^2-1}\left[\cos\left(\frac{(n+1)\pi}{2} + (n+1)\pi\right) - \cos\frac{(n+1)\pi}{2}\right] \\
= &\quad \frac{6}{\pi 6^n}\frac{1}{n^2-1}\left((-1)^{n+1} - 1\right)\cos\frac{(n+1)\pi}{2} \\
= &\quad \begin{cases} \frac{6}{\pi 6^{2k}}\frac{1}{(2k)^2-1}\left((-1)^{2k+1} - 1\right)\cos\frac{(2k+1)\pi}{2} = 0 & n = 2k \\ \frac{6}{\pi 6^{2k+1}}\frac{1}{(2k+1)^2-1}\left((-1)^{2k+2} - 1\right)\cos\frac{(2k+2)\pi}{2} = 0 & n = 2k+1 \end{cases}
\end{aligned}$$

So collecting the results above, we get

$$\alpha_0 = -\frac{12}{\pi}$$

$$\alpha_n = \begin{cases} \frac{1}{2} & n = 1 \\ \frac{(-1)^k 12}{6^{2k}[(2k)^2 - 1]\pi} & n = 2k, k = 1, 2... \\ 0 & n = 2k+1, k = 1, 2... \end{cases}$$

$$\beta_n = 0$$

Hence the solution is

$$u(r, \theta) = -\frac{6}{\pi} + \frac{r\cos\theta}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k 12}{6^{2k}[(2k)^2 - 1]\pi} r^n \cos(2k\theta)$$

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).