Solution 13

1. The characteristic equations and the parametric initial conditions are

$$\begin{array}{lll} x_t(t,s) = x^2, & y_t(t,s) = -(y^2+1) & u_t(t,s) = ux^3 \\ x(0,s) = 1 & y(0,s) = s & u(0,s) = 2 \end{array}$$

   Solve the equations:

$$\begin{aligned} & x_t(t,s) = x^2 \\ \Rightarrow\quad & \left(\frac{1}{x}(t,s)\right)_t = -1 \\ \Rightarrow\quad & \frac{1}{x(t,s)} = -t + \frac{1}{x(0,s)} = 1-t \\ \Rightarrow\quad & x(t,s) = \frac{1}{1-t} \end{aligned}$$

   and

$$\begin{aligned} & u_t(t,s) = ux^3 \\ \Rightarrow\quad & (\log u(t,s))_t = x^3 = \frac{1}{(1-t)^3} = \left(\frac{1}{2(1-t)^2}\right)_t \\ \Rightarrow\quad & \log u(t,s) = \log u(0,s) + \left.\frac{1}{2(1-t)^2}\right|_0^t = \log 2 + \frac{1}{2(1-t)^2} - \frac{1}{2} \\ \Rightarrow\quad & u(t,s) = 2e^{\frac{1}{2(1-t)^2}-\frac{1}{2}} \end{aligned}$$

   Without solving the equation of y, we have already obtained the solution

$$u(x,y) = 2e^{\frac{x^2}{2}-\frac{1}{2}}$$

2. The constant function $u_1(x,y) = 1$ is a solution for the problem.

   We know that the fundamental solution $w(x,y) = -\frac{1}{2\pi}\log\sqrt{x^2+y^2}$ is harmonic in any domain excluding the origin (see p. 177-178 in [PR]). Then $u_2(x,y) = 1 + \log\sqrt{x^2+y^2}$ is harmonic in $\mathbb{R}^2\backslash D$ and $u_2(x,y) = 1$ for $(x,y) \in \partial D = \{x^2+y^2=1\}$, hence u_2 is another solution for the problem, different from u_1.

   So the solution of the problem is not unique.

a. Firstly, we prove the following lemma

Lemma. *Let v be a C^2 function satisfying $v_t - k\Delta v - yv_x + x^3 v_y < 0$ in Q_T. Then v has no local maximum in $Q_T \backslash \partial_P Q_T$. Moreover, v achieves its maximum in $\partial_P Q_T$.*

Proof of the lemma. If v has a local maximum at some $q = (x_q, y_q, t_q) \in Q_T \backslash \partial_P Q_T$. Since (x_q, y_q) is in the interior of D, i.e. $(x_q, y_q) \in \{x^2 + y^2 < 1\}$, we have $\Delta v(q) \leqslant 0$ and $v_x(q) = v_y(q) = 0$. There are two possibilities for $q = (x, y, t)$:

$t = T$: Then $v(q)_t \geqslant 0$, hence $v_t - k\Delta v - yv_x + x^3 v_y \geqslant 0$, contradicting the assumption on v.

$0 < t < T$: Then $v(q)_t = 0$, hence $v_t - k\Delta v - yv_x + x^3 v_y \geqslant 0$, again contradicting the assumption on v.

Hence v cannot have a local maximum in $Q_T \backslash \partial_P Q_T$. □

We consider the function $u_\varepsilon = u - \varepsilon t$ for any $\varepsilon > 0$. It satisfies

$$u_{\varepsilon t} - k u_\varepsilon - y u_{\varepsilon x} + x^3 u_{\varepsilon y} = -\varepsilon < 0.$$

By the lemma u_ε achieves its maximum in $\partial_P Q_T$,

$$\max_{Q_T} u_\varepsilon = \max_{\partial_P Q_T} u_\varepsilon.$$

Since

$$\max_{Q_T} u_\varepsilon = \max_{Q_T}(u - \varepsilon t) \geqslant \max_{Q_T} u - \varepsilon T,$$

$$\max_{\partial_P Q_T} u_\varepsilon = \max_{\partial_P Q_T} u - \varepsilon t \leqslant \max_{\partial_P Q_T} u,$$

it follows that

$$\max_{Q_T} u - \varepsilon T \leqslant \max_{\partial_P Q_T} u.$$

Since ε can be made arbitrary small, we obtain

$$\max_{Q_T} u \leqslant \max_{\partial_P Q_T} u,$$

which means that u achieves its maximum at $\partial_P Q_T$.

4. We rewrite the equation as following

$$u_t + \left(\frac{1}{3}u^3\right)_x = 0.$$

We assume the solution takes the form

$$u(x,t) = \begin{cases} 3 & x < \gamma(t), \\ 1 & x > \gamma(t). \end{cases}$$

Then by the Rankine-Hugoniot condition,

$$\gamma_t(t) = \frac{\frac{1}{3}u_+^3 - \frac{1}{3}u_-^3}{u_+ - u_-} = \frac{\frac{1}{3} - \frac{3^3}{3}}{1-3} = \frac{13}{3}$$

Hence $\gamma(t) = \frac{13}{3}t$. The weak solution of the problem is

$$u(x,t) = \begin{cases} 3 & x < \dfrac{13}{3}t \\ 1 & x > \dfrac{13}{3}t. \end{cases}$$

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).