Solution 2

1) The characteristic equations and the parametric initial conditions are

$$x_t(t,s) = 1, \quad y_t(t,s) = 1, \quad u_t(t,s) = 2u + 1,$$
$$x(0,s) = s, \quad y(0,s) = 0, \quad u(0,s) = 0.$$

Solve the above equations to obtain the characteristic curves:

$$x(t,s) = t + s, \quad y(t,s) = t, \quad u(t,s) = \frac{e^{2t} - 1}{2}.$$

Invert the transformation $(x(t,s), y(t,s))$,

$$t = y, \quad s = x - y.$$

Substituting to u, we get

$$u(x,y) = \frac{e^{2y} - 1}{2}.$$

2) The characteristic equations and the parametric initial conditions are

$$x_t(t,s) = x, \quad y_t(t,s) = x + y, \quad u_t(t,s) = 1,$$
$$x(0,s) = 1, \quad y(0,s) = s, \quad u(0,s) = s.$$

Solve the above equations to obtain the characteristic curves:

$$x(t,s) = e^t, \quad y(t,s) = (t+s)e^t, \quad u(t,s) = t + s.$$

Invert the transformation $(x(t,s), y(t,s))$,

$$t = \log x, \quad s = \frac{y}{x} - \log x, \quad x > 0.$$

Substituting to u, we get

$$u(x,y) = \frac{y}{x}, \quad x > 0.$$

3) The characteristic equations and the parametric initial conditions are

$$x_t(t,s) = 1, \quad y_t(t,s) = 2, \quad u_t(t,s) = e^{u(t,s)} \Leftrightarrow (e^{-u(t,s)})_t = -1,$$
$$x(0,s) = 0, \quad y(0,s) = s, \quad u(0,s) = 1.$$

Solve the above equations to obtain the characteristic curves:

$$x(t,s) = t, \quad y(t,s) = 2t + s, \quad e^{-u(t,s)} = -t + e^{-1} \Leftrightarrow u(t,s) = -\log(e^{-1} - t).$$

Invert the transformation $(x(t,s), y(t,s))$,

$$t = x, \quad s = y - 2x.$$

Substituting to u, we get

$$u(x,y) = -\log(e^{-1} - x), \quad x < e^{-1}.$$

4) The characteristic equations and the parametric initial conditions are

$$x_t(t,s) = x, \quad y_t(t,s) = y, \quad u_t(t,s) = -u,$$
$$x(0,s) = \cos s, \quad y(0,s) = \sin s, \quad u(0,s) = 1.$$

Solve the above equations to obtain the characteristic curves:

$$x(t,s) = e^t \cos s, \quad y(t,s) = e^t \sin s, \quad u(t,s) = e^{-t}.$$

Then we get

$$u(x,y) = \frac{1}{\sqrt{x^2 + y^2}}, \quad (x,y) \neq (0,0).$$