Solution 3

1) a) The characteristic equations and the parametric initial conditions are

$$x_t(t,s) = y, \quad y_t(t,s) = -x, \quad u_t(t,s) = 0,$$
$$x(0,s) = s, \quad y(0,s) = 0, \quad u(0,s) = s^2.$$

Solve the above equations to obtain the characteristic curves:

$$x(t,s) = s\cos t, \quad y(t,s) = -s\sin t, \quad u(t,s) = s^2,$$

so

$$u(x,y) = x^2 + y^2, \quad y \geqslant 0,$$

is a solution for the initial condition $u(x,0) = x^2$.

b) Similar arguments as in **a)**, we have the characteristic curves

$$x(t,s) = s\cos t, \quad y(t,s) = -s\sin t, \quad u(t,s) = s, \quad s > 0, -\pi \leqslant t \leqslant 0.$$

The projections of characteristic curves on (x,y) plane intersects the initial curve $\{(x,0)|x \in \mathbb{R}\}$ twice for $s > 0$ at $(s,0)$ and $(-s,0)$, which are projections of

$$(x(0,s), y(0,s), u(0,s)) = (s,0,s)$$

and

$$(x(-\pi,s), y(-\pi,s), u(-\pi,s)) = (-s,0,s).$$

If there is a solution $u(x,y)$ for the equation on $y \geqslant 0$, from the characteristic curves above, we get

$$(x(-\pi,s), y(-\pi,s), u(-\pi,s)) = (-s,0,s)$$

would be a point in the image of $u(x,y)$ in (x,y,u) space, which means $u(-s,0) = s$, contradicting $u(-s,0) = -s$. So for the initial condition $u(x,0) = x, x \in \mathbb{R}$, the equation is not solvable.

c) Similar arguments as in **b)**, we have the characteristic curves

$$x(t,s) = s\cos t, \quad y(t,s) = -s\sin t, \quad u(t,s) = s, \quad s > 0, -\pi \leqslant t \leqslant 0.$$

so

$$u(x,y) = \sqrt{x^2 + y^2}, \quad y \geqslant 0,$$

is a solution for the initial condition $u(x,0) = x, x > 0$.

2) a) The characteristic equations and the parametric initial conditions are

$$x_t(t,s) = x, \quad y_t(t,s) = y, \quad u_t(t,s) = \frac{1}{\cos u} \Leftrightarrow (\sin u)_t = 1,$$
$$x(0,s) = s^2, \quad y(0,s) = \sin s, \quad u(0,s) = 0.$$

Solve the above equations to obtain the characteristic curves:

$$x(t,s) = s^2 e^t, \quad y(t,s) = e^t \sin s, \quad \sin u(t,s) = t.$$

Invert the transformation $(x(t,s), u(t,s))$,

$$t = \sin u, \quad s = \sqrt{\frac{x}{e^{\sin u}}}.$$

Substituting to y, we get

$$y = e^{\sin u} \sin\left(\sqrt{\frac{x}{e^{\sin u}}}\right)$$

b) Let us check the transversality condition, the Jacobian

$$J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} x & y \\ 2s & \cos s \end{vmatrix} = \begin{vmatrix} s^2 & \sin s \\ 2s & \cos s \end{vmatrix} = s^2 \cos s - 2s \sin s.$$

The transversality condition holds if $s^2 \cos s - 2s \sin s \neq 0 \Leftrightarrow \tan s \neq \frac{s}{2}$. From the function images of $\tan s$ and $\frac{s}{2}$, we know that $\tan s = \frac{s}{2}$ has infinity discrete solutions, so for any domain of s values excluding these solutions, there exists a unique solution, for example $s \in (0, \frac{\pi}{2})$, since $\tan s > \frac{s}{2}$ for $s \in (0, \frac{\pi}{2})$.

3) a) The characteristic equations and the parametric initial conditions are

$$x_t(t,s) = x, \quad y_t(t,s) = x^2 + y, \quad u_t(t,s) = 1 - (\frac{y}{x} - x)u,$$
$$x(0,s) = 1, \quad y(0,s) = s+1, \quad u(0,s) = 0.$$

Solve the above equations to obtain the characteristic curves:
from the equation of x, we have

$$x(t,s) = e^t,$$

then substituting x into the equation of y, we get

$$y_t(t,s) = y + e^{2t}.$$

To solve the above equation, we refer to **Example 2.1** on page 24 in [PR] (or the appendix **A.3 Elementary ODEs** 1) on page 362 in [PR]):

$$\begin{aligned} y(t,s) &= e^t\left[\int_0^t e^{-\xi}e^{2\xi}\mathrm{d}\xi + c\right] \\ &= e^{2t} + (c-1)e^t, \end{aligned}$$

and from the initial condition $y(0,s) = s+1$, we get

$$y(t,s) = e^{2t} + se^t.$$

Then we substitute x and y in the equation of u,

$$u_t(t,s) = 1 - (\frac{y}{x} - x)u = 1 - su(t,s) \quad \Rightarrow \begin{cases} u(t,s) = \dfrac{1-e^{-st}}{s} & s \neq 0, \\ u(t,0) = t. \end{cases}$$

The case $s \neq 0$:

$$\frac{u_t}{1-su} = 1 \Leftrightarrow \left(-\frac{1}{s}\ln(1-su)\right)_t = 1 \Leftrightarrow -\frac{1}{s}\ln(1-su) = c + t \Leftrightarrow u = \frac{1-e^{-st-sc}}{s}.$$

the initial condition $u(0,s) = 0$ implies $c = 0$, hence

$$u = \frac{1-e^{-st}}{s}$$

Invert the transformation $(x(t,s), y(t,s))$,

$$t = \ln x, \quad s = \frac{y}{x} - x.$$

Substituting to u, we get

$$u(x,y) = \begin{cases} \dfrac{x}{x^2-y}(x^{x-\frac{y}{x}} - 1), & x^2 - y \neq 0, \\ \ln x, & x^2 - y = 0. \end{cases}$$

b) The Jacobian

$$J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} x(0,s) & x^2(0,s) + y(0,s) \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1+y \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so the transversality condition holds everywhere.

c) From the solution of the characteristic equations, we know

$$x(t,s) = e^t, \quad y(t,s) = se^t + e^{2t},$$

hence the projection of characteristic curves on (x, y) plane are the images of functions $y = sx + x^2$.
The projection on (x, y) plane of characteristic curves emanating from $(1, 2, 0)$ is the image of function$y = x + x^2 = (x + \frac{1}{2})^2 - \frac{1}{4}$.
The projection on (x, y) plane of characteristic curves emanating from $(1, 0, 0)$ is the image of function$y = -x + x^2 = (x - \frac{1}{2})^2 - \frac{1}{4}$.
Draw the corresponding pictures..... (See the pictures on last page.)

4) a) The characteristic equations are

$$x_t(t,s) = y, \quad y_t(t,s) = u, \quad u_t(t,s) = x,$$

implying

$$(x + y + u)_t(t,s) = (x + y + u)(t,s).$$

From the initial condition, we have $(x + y + u)(0, s) = 0$, so $(x + y + u)(t, s) = 0$. The solution is $u = -x - y$.

b) The characteristic equations and initial conditions are

$$\begin{aligned} x_t(t,s) = y, \quad y_t(t,s) = u, \quad u_t(t,s) = x, \\ x(0,s) = s, \quad y(0,s) = s, \quad u(0,s) = s. \end{aligned}$$

Let us examine the transversality condition. The Jacobian

$$J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} y(0,s) & u(0,s) \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} s & s \\ 1 & 1 \end{vmatrix} = 0.$$

So the transversality condition fails for $-\infty < s < \infty$. By the existence-uniqueness theorem, we need to examine whether the initial curve $(x(0, s), y(0, s), u(0, s))$ agrees with some characteristic curve. The characteristic equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = y(t), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}(t) = u(t), \\ \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = x(t), \end{cases}$$

have solutions $(x(t), y(t), u(t)) = (e^t, e^t, e^t)$ and $(x(t), y(t), u(t)) = (-e^t, -e^t, -e^t)$. These characteristic curves agree with the initial curve, hence the Cauchy problem has infinite solutions.

c) Let $(x(t), y(t), u(t))$ be a characteristic curve.

$$
\begin{aligned}
\tfrac{\mathrm{d}w_1}{\mathrm{d}t}(t) &= w_1(t),\\
\tfrac{\mathrm{d}w_2}{\mathrm{d}t}(t) &= 2x(t)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + 2y(t)\frac{\mathrm{d}y}{\mathrm{d}t}(t) + 2u(t)\frac{\mathrm{d}u}{\mathrm{d}t}(t)\\
&= 2x(t)y(t) + 2y(t)u(t) + 2u(t)x(t) = 2w_3(t),\\
\tfrac{\mathrm{d}w_3}{\mathrm{d}t}(t) &= y\frac{\mathrm{d}x}{\mathrm{d}t}(t) + x\frac{\mathrm{d}y}{\mathrm{d}t}(t) + x\frac{\mathrm{d}u}{\mathrm{d}t}(t) + u\frac{\mathrm{d}x}{\mathrm{d}t}(t) + y\frac{\mathrm{d}u}{\mathrm{d}t}(t) + u\frac{\mathrm{d}y}{\mathrm{d}t}(t)\\
&= y^2(t) + x(t)u(t) + x^2(t) + u(t)y(t) + u^2(t) + y(t)x(t) = w_2(t) + w_3(t),
\end{aligned}
$$

so

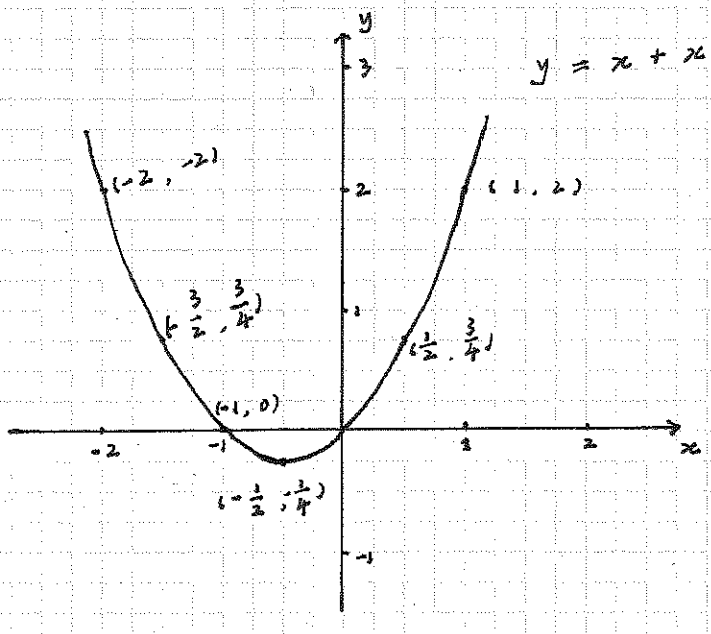
$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}w_1(w_2 - w_3)(t) &= \frac{\mathrm{d}w_1}{\mathrm{d}t}(t)(w_2(t) - w_3(t)) + w_1(t)(\frac{\mathrm{d}w_2}{\mathrm{d}t}(t) - \frac{\mathrm{d}w_3}{\mathrm{d}t}(t))\\
&= w_1(w_2 - w_3)(t) + w_1(2w_3 - w_2 - w_3)(t) = 0.
\end{aligned}
$$

Hence $w_1(w_2 - w_3)$ is constant along each characteristic curve.

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).

$$y = x + x^2 = (x + \tfrac{1}{2})^2 - \tfrac{1}{4}.$$



$$y = -x + x^2 = (x - \tfrac{1}{2})^2 - \tfrac{1}{4}.$$