Solution 4

1. a) Let $h(s) = u(s,0)$, since the initial condition $h(x)$ is differentiable and nondecreasing, i.e. $h'(x) \geqslant 0$, we have the formula for the critical time (see equation (2.47) on p.43 in [PR]),

$$y_c = -\frac{1}{h'(x)} < 0 \quad \forall x \in \mathbb{R},$$

hence the Cauchy problem has a differentiable solution for all positive time $y > 0$.

b) Let $h(s) = u(s,0)$, since $h(s)$ is decreasing in $[\frac{-\pi}{2}, \frac{\pi}{2}]$, i.e. $h'(s) < 0$, the solution will become non-differentiable. We have the formula (2.46) in [PR],

$$u_x = \frac{h'}{1 + yh'},$$

then the solution's derivate blows up at the time $y = -\frac{1}{h'(s)}$, the critical time y_c is the infimum of $\{-\frac{1}{h'(s)}\}$.

$$h'(s) = \begin{cases} 0 & x \leqslant -\dfrac{\pi}{2}, \\ -\cos x & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}, \\ 0 & x \geqslant \dfrac{\pi}{2}. \end{cases}$$

$h'(s)$ has infimum -1 at $s = 0$, so

$$y_c = \inf\{-\frac{1}{h'(s)}\} = -\frac{1}{h'(0)} = 1.$$

c) The discontinuity moves with a speed $\frac{1}{2}(u_- + u_+) = \frac{1}{2}(1 - 2) = -\frac{1}{2}$, therefore

$$u(x,y) = \begin{cases} 1 & x < 1 - \dfrac{y}{2}, \\ -2 & x > 1 - \dfrac{y}{2}. \end{cases}$$

is a weak solution of the Cauchy problem

2. a) The characteristic equations and the parametric initial conditions are

$$y_t(t,s) = 1, \quad x_t(t,s) = u^2, \quad u_t(t,s) = 0,$$

$$y(0,s) = 0, \quad x(0,s) = s, \quad u(0,s) = \begin{cases} 1 & s \leqslant 0, \\ \sqrt{1 - \dfrac{s}{\alpha}} & 0 < s < \alpha, \\ 0 & s \geqslant \alpha. \end{cases}$$

Solve the equations to obtain the characteristics

$$(x,y,u)(t,s) = (s + u^2(0,s)t, t, u(0,s)) = \begin{cases} (s+t, t, 1) & s \leqslant 0, \\ (s + (1 - \dfrac{s}{\alpha})t, t, \sqrt{1 - \dfrac{s}{\alpha}}) & 0 < s < \alpha, \\ (s, t, 0) & s \geqslant \alpha. \end{cases}$$

Two ways to proceed:

1st. Invert the transformation $(x(t,s), y(t,s))$

$s \leqslant 0$

$$(x,y) = (s+t, t) \Rightarrow (t,s) = (y, x-y), \quad s \leqslant 0 \Rightarrow x - y \leqslant 0 \Leftrightarrow x \leqslant y$$

$0 < s < \alpha$

$$(x,y) = (s + (1 - \frac{s}{\alpha})t, t) \Rightarrow (t,s) = (y, \alpha\frac{x-y}{\alpha - y}),$$

$$0 < s < \alpha \Rightarrow 0 < \alpha\frac{x-y}{\alpha-y} < \alpha \Leftrightarrow y < x < \alpha \text{ or } \alpha < x < y$$

$s \geqslant \alpha$

$$(x,y) = (s,t)$$

Then for $0 < y < \alpha$, we have

$$(t,s) = \begin{cases} (y, x-y) & x \leqslant y, \\ (y, \alpha\dfrac{x-y}{\alpha-y}) & y < x < \alpha, \\ (y, x) & x \geqslant \alpha. \end{cases}$$

Substitutie to $u(t,s)$,

$$u(x,y) = \begin{cases} 1 & x \leqslant y, \\ \sqrt{\dfrac{x-\alpha}{y-\alpha}} & y < x < \alpha, \\ 0 & x \geqslant \alpha. \end{cases}$$

The solution becomes discontinuous at $y = \alpha$.

2nd.

$$u = u(0, s = x - u^2 y) = \begin{cases} 1 & x \geqslant y, \\ \sqrt{1 - \dfrac{x - u^2 y}{\alpha}} & y < x < \alpha, \\ 0 & x \geqslant \alpha. \end{cases}$$

Solve u to get the same result.

b) Two ways to argue that the solution will be sigular:

1st. The projections of characteristic curves (x, y)-plane are the lines

$$x = s + u^2(0, s)y = \begin{cases} s + y & s \leqslant 0, \\ s + (1 - \dfrac{s}{\alpha})y & 0 < s < \alpha, , \\ s & s \geqslant \alpha, \end{cases}$$

for the s-values $0 \leqslant s \leqslant \alpha$, these lines will collide at a finite y, actually they collide at one point $(x, y) = (\alpha, \alpha)$. However the function $u(x, y)$ preserve its initial value $u(x, 0)$ along these lines, hence we can't solve the Cauchy problem beyond $y = \alpha$. The critical time $y_c = \alpha$.

2nd. From the explicit form of the solution $u(x, y)$, the solution is continuous for $y < \alpha$. When $y = \alpha$,

$$u(x, \alpha) = \begin{cases} 1 & x < \alpha, \\ 0 & x > \alpha, \end{cases}$$

which is discontinuous in x. So we can't solve the Cauchy problem beyond $y = \alpha$. The critical time $y_c = \alpha$.

c) See the 1st argument of b).

$$\ell_0 : x = y; \quad \ell_{\frac{\alpha}{2}} : x = \frac{\alpha}{2} + \frac{y}{2}; \quad \ell_\alpha : x = \alpha.$$

Direct calculations show they intersect at (α, α).

3. a) Rewrite the equation in the form

$$u_y + \frac{1}{3}(u^3)_x = 0,$$

and integrate (with respect to x, for a fixed y) over an arbitrary interval $[a, b]$ to obtain

$$\partial_y \int_a^b u(\xi, y)\mathrm{d}\xi + \frac{1}{3}[u^3(b, y) - u^3(a, y)] = 0.$$

b) We write the weak formulation in the form

$$\partial_y \left[\int_a^{\gamma(y)} u(\xi, y) \mathrm{d}\xi + \int_{\gamma(y)}^b u(\xi, y) \mathrm{d}\xi \right] + \frac{1}{3}[u^3(b, y) - u^3(a, y)] = 0$$

Differentiating the integrals with respect to y and using the PDE itself leads to

$$\gamma_y(y) u_-(y) - \gamma_y(y) u_+(y) - \frac{1}{3} \left[\int_a^{\gamma(y)} (u^3(\xi, y))_\xi \mathrm{d}\xi + \int_{\gamma(y)}^b (u^3(\xi, y))_\xi \mathrm{d}\xi \right]$$
$$+ \frac{1}{3}[u^3(b, y) - u^3(a, y)] = 0$$
$$\Rightarrow$$
$$\gamma_y(y)[u_-(y) - u_+(y)] - \frac{1}{3} \left[u_-^3(y) - u^3(a, y) + u^3(b, y) - u_+^3(y) \right]$$
$$+ \frac{1}{3}[u^3(b, y) - u^3(a, y)] = 0.$$

We get

$$\gamma_y(y) = \frac{\frac{1}{3}[u_+^3(y) - u_-^3(y)]}{u_+(y) - u_-(y)}.$$

c) The solution of the Cauchy problem in **Ex 2** at $y_c = \alpha$

$$u(x, \alpha) = \begin{cases} 1 & x < \alpha, \\ 0 & x > \alpha, \end{cases}$$

implies

$$u_-(\alpha) = 1, u_+(\alpha) = 0,$$

then b) implies that the discontinuity moves with a speed $\frac{1}{3}$. Therefore the following weak solution is compatible with the integral balance for $y > \alpha$:

$$u(x, y) = \begin{cases} 1 & x < \alpha + \frac{1}{3}(y - \alpha), \\ 0 & x > \alpha + \frac{1}{3}(y - \alpha). \end{cases}$$

4. a)

$$L[u](x, y) = u_{xx} + 2u_{xy} + [1 - q(y)] u_{yy},$$

and $a = 1, b = 1, c = 1 - q(y)$, hence

$$\delta(L)(x, y) = b^2 - ac = 1^2 - [1 - q(y)] = \begin{cases} -1 & y < -1, \\ 0 & |y| \leqslant 1, \\ 1 & y > 1. \end{cases}$$

So the equation is hyperbolic on $\{y > 1\}$, parabolic on $\{|y| \leqslant 1\}$, elliptic on $\{y < -1\}$.

b) • $y > 1$:

$$L[u] = u_{xx} + 2u_{xy}.$$

Consider a nonsingular linear transformation

$$\xi = \xi_x x + \xi_y y, \quad \eta = \eta_x x + \eta_y y.$$

and $\omega(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, then ω satisfies the equation

$$\ell[\omega] = A\omega_{\xi\xi} + 2B\omega_{\xi\eta} + C\omega_{\eta\eta} = 0$$

where

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}$$

To obtain the canonical form, we need to solve the equation

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = \xi_x^2 + 2\xi_x\xi_y = 0,$$
$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \eta_x^2 + 2\eta_x\eta_y = 0$$

We can take $\xi_x = 2, \xi_y = -1, \eta_x = 0, \eta_y = 1$, i.e. the transformation

$$\xi = 2x - y, \quad \eta = y$$

transforms the equation to canonical form

$$\ell[\omega] = 4\omega_{\xi\eta} = 0$$

• $|y| < 1$:

$$L[u] = u_{xx} + 2u_{xy} + u_{yy}.$$

Consider a nonsingular linear transformation

$$\xi = \xi_x x + \xi_y y, \quad \eta = \eta_x x + \eta_y y.$$

and $\omega(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, then ω satisfies the equation

$$\ell[\omega] = A\omega_{\xi\xi} + 2B\omega_{\xi\eta} + C\omega_{\eta\eta} = 0$$

where

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}$$

To obtain the canonical form, we need to solve the equation

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \eta_x^2 + 2\eta_x\eta_y + \eta_y^2 = 0$$

We can take $\xi_x = 1, \xi_y = 1, \eta_x = 1, \eta_y = -1$, i.e. the transformation

$$\xi = x + y, \quad \eta = x - y$$

transforms the equation to canonical form

$$\ell[\omega] = 4\omega_{\xi\xi} = 0$$

- $y < -1$:

$$L[u] = u_{xx} + 2u_{xy} + 2u_{yy}.$$

Consider a nonsingular linear transformation

$$\xi = \xi_x x + \xi_y y, \quad \eta = \eta_x x + \eta_y y.$$

and $\omega(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, then ω satisfies the equation

$$\ell[\omega] = A\omega_{\xi\xi} + 2B\omega_{\xi\eta} + C\omega_{\eta\eta} = 0$$

where

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}$$

To obtain the canonical form, we need to solve the equation

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = \xi_x^2 + 2\xi_x\xi_y + 2\xi_y = 1,$$
$$B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = \xi_x\eta_x + (\xi_x\eta_y + \xi_y\eta_x) + 2\xi_y\eta_y = 0,$$
$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \eta_x^2 + 2\eta_x\eta_y + 2\eta_y^2 = 1$$

We can take $\xi_x = 1, \xi_y = 0, \eta_x = 1, \eta_y = -1$, i.e. the transformation

$$\xi = x, \quad \eta = x - y$$

transforms the equation to canonical form

$$\ell[\omega] = \omega_{\xi\xi} + \omega_{\eta\eta} = 0$$

c) For the hyperbolic case $\{y > 1\}$,

$$L[u] = u_{xx} + 2u_{xy},$$

so

$$a = 1, \quad b = 1, \quad c = 0,$$

and the characteristic equations are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b + \sqrt{b^2 - ac}}{a} = 2,$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b - \sqrt{b^2 - ac}}{a} = 0$$

so the characteristics are the straight lines $\{y = 2x + s\}$ and $\{y = s\}$. Draw the pictures.......

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).