Solution 5

1. a) By the d'Alembert's formula,

$$u(x,t) = \frac{\sin(x+ct)+\sin(x-ct)}{2} = \sin(x)\cos(ct), \quad -\infty < x < \infty,\ t > 0.$$

The function $\frac{\sin(x-ct)}{2}$ is the forward wave and the function $\frac{\sin(x+ct)}{2}$ is the backward wave.

b)

$$u(x, t + \frac{2\pi}{c}) = \sin(x)\cos(ct+2\pi) = \sin(x)\cos(ct) = u(x,t).$$

2. Using the d'Alembert's formula, we find

$$P(10,t) = \frac{P(10-4t,0)+P(10+4t,0)}{2} + \frac{1}{8}\int_{10-4t}^{10+4t} P_t(s,0)\mathrm{d}s$$

Since $P_t(s,0)$ is a step function

$$P_t(x,0) = \begin{cases} 1 & |x| \leqslant 1 \Leftrightarrow -1 \leqslant x \leqslant 1, \\ 0 & |x| > 1 \Leftrightarrow x \leqslant -1 \vee x \geqslant 1, \end{cases}$$

we calculate the integral $\frac{1}{8}\int_{10-4t}^{10+4t} P_t(s,0)\mathrm{d}s$ in three cases

$1 < 10-4t$

$$\frac{1}{8}\int_{10-4t}^{10+4t} P_t(s,0)\mathrm{d}s = \frac{1}{8}\int_{10-4t}^{10+4t} 0\mathrm{d}s,$$

$-1 \leqslant 10-4t \leqslant 1 < 10+4t$

$$\frac{1}{8}\int_{10-4t}^{10+4t} P_t(s,0)\mathrm{d}s = \frac{1}{8}\int_{1}^{10+4t} 0\mathrm{d}s + \frac{1}{8}\int_{10-4t}^{1} 1\mathrm{d}s = \frac{1-(10-4t)}{8} = \frac{4t-9}{8},$$

$10-4t < -1 < 1 < 10+4t$

$$\frac{1}{8}\int_{10-4t}^{10+4t} P_t(s,0)\mathrm{d}s = \frac{1}{8}\int_{10-4t}^{-1} 0\mathrm{d}s + \frac{1}{8}\int_{-1}^{1} 1\mathrm{d}s + \frac{1}{8}\int_{1}^{10+4t} 0\mathrm{d}s = \frac{1}{4}.$$

So

$$P(10,t) = \begin{cases} 0 & 10-4t > 1 \Rightarrow t \in (0, \frac{9}{4}), \\ 5 + (\frac{t}{2} - \frac{9}{8}) & -1 \leqslant 10-4t \leqslant 1 \Rightarrow t \in [\frac{9}{4}, \frac{11}{4}], \\ \frac{1}{4} & 10-4t < -1 \Rightarrow t \in (\frac{11}{4}, \infty). \end{cases}$$

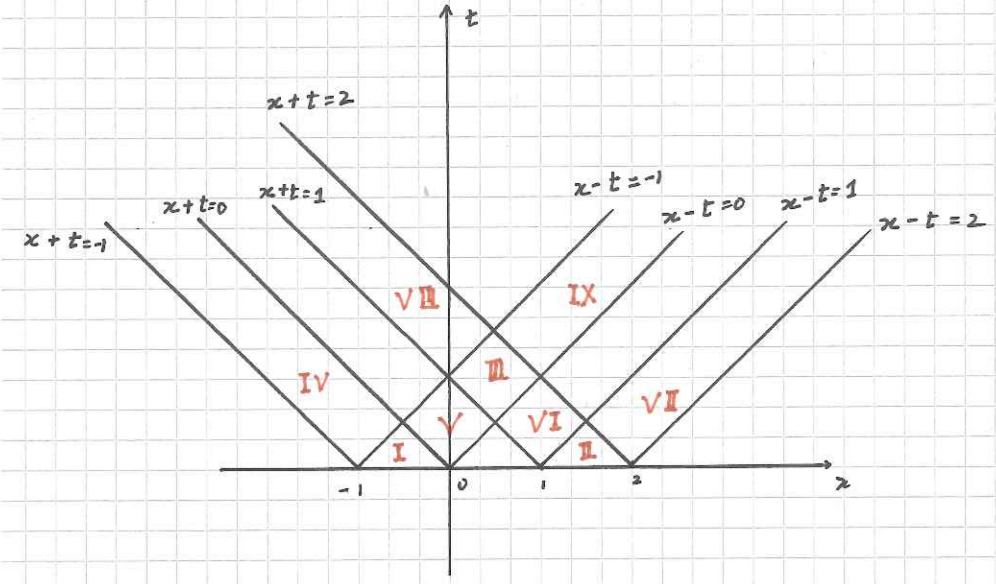
The pressure at the building achieve the maximal at $t_0 = \frac{11}{4}$. $P(10, \frac{11}{4}) = 5\frac{1}{4} < 6$, so the building won't collapse.

3.

$$\begin{aligned} &\tilde{u}(x,T) = u(x,0) = f(x), \\ &\tilde{u}_t(x,T) = \partial_t|_{t=T}[u(x,T-t)] = -u_t(x,0) = -g(x), \\ &\tilde{u}_{tt}(x,t) = [u(x,T-t)]_{tt} = -[u_t(x,T-t)]_t = u_{tt}(x,T-t), \\ &\tilde{u}_{xx} = [u(x,T-t)]_{xx} = u_{xx}(x,T-t), \\ &\tilde{u}_{tt}(x,t) - c^2\tilde{u}_{xx}(x,t) = u_{tt}(x,T-t) - c^2u_{xx}(x,T-t) = 0. \end{aligned}$$

4. The points where the initial data are not smooth are $(-1,0), (0,0), (1,0), (2,0)$. We draw the characteristic curves through them and divide the upper plane $t > 0$ by these lines. Using the d'Alembert's formula, we have

$$u(x,t) = \frac{u(x-t,0) + u(x+t,0)}{2} = \begin{cases} 1 & (x,t) \text{ contained in the regions I, II, III} \\ \frac{1}{2} & (x,t) \text{ contained in the regions IV, V, VI, VII, VIII, IX} \\ 0 & \text{otherwise.} \end{cases}$$



t
x + t = 2
x + t = 0
x + t = 1
x + t = -1
x - t = -1
x - t = 0
x - t = 1
x - t = 2
VIII
IX
IV
III
V
VI
VII
I
II
-1
0
1
2
x