Solution 7

1. a) Find solutions of the wave equation with Dirichlet boundary conditions

$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < \pi, \quad t > 0, \\ &u(0,t) = u(\pi,t) = 0 && t \geqslant 0, \end{aligned}$$

having the special form

$$u(x,t) = X(x)T(t).$$

Substituting $u(x,t)$ to the wave equation, we obtain

$$XT_{tt} = X_{xx}T \Leftrightarrow \frac{X_{xx}}{X} = \frac{T_{tt}}{T}.$$

It follows there exists a constant λ such that

$$\frac{X_{xx}}{X} = \frac{T_{tt}}{T} = -\lambda,$$

which implies that

$$\begin{aligned} \frac{\mathrm{d}^2X}{\mathrm{d}x^2} &= -\lambda X \quad 0 < x < \pi, \\ \frac{\mathrm{d}^2T}{\mathrm{d}x^2} &= -\lambda T \quad t > 0. \end{aligned}$$

The Dirichlet boundary conditions imply

$$X(0)T(t) = X(\pi)T(t) = 0, \quad t > 0,$$

since u is nontrivial it follows that

$$X(0) = X(\pi) = 0,$$

then the function X is a solution of the eigenvalue problem

$$\begin{aligned} &\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \lambda X = 0 && 0 < x < \pi, \\ &X(0) = X(\pi) = 0. \end{aligned}$$

Following the similar arguments in p.110-111 in [PR], we can show that the eigenvalues are

$$\lambda_n = n^2, \quad n = 1, 2, ...$$

and the associated eigenfunctions are

$$X_n(x) = \sin nx,$$

and it is uniquely determined up to a multiplicative factor. Consider the equation for T with $\lambda = \lambda_n$, we get the corresponding solution T_n

$$T_n = \gamma_n \cos nt + \delta_n \sin nt, \quad n = 1, 2...$$

Thus , the product solutions of the initial boundary value problem are given by

$$u_n(x,t) = \sin nx\,(A_n \cos nt + B_n \sin nt)$$

Applying the superposition principle, the expansion

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx\,(A_n \cos nt + B_n \sin nt)$$

is a generalized solution of the problem. The constants A_n and B_n are determined from the initial conditions,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx,$$
$$u_t(x,0) = \sum_{n=1}^{\infty} nB_n \sin nx.$$

Consider the initial conditions in this problem

$$\begin{aligned} u(x,0) &= \sin^3 x = (\frac{1}{2} - \frac{\cos 2x}{2}) \sin x \\ &= \frac{1}{2} \sin x - \frac{1}{2} \cos 2x \sin x \\ &= \frac{1}{2} \sin x - \frac{1}{4}(\sin 3x - \sin x) \\ &= \frac{3}{4} \sin x - \frac{1}{4} \sin 3x, \\ u_t(x,0) &= \sin 2x, \end{aligned}$$

thus,

$$A_1 = \frac{3}{4}, \quad A_3 = -\frac{1}{4}, \quad A_n = 0\, \forall n \neq 1, 3,$$
$$B_2 = \frac{1}{2}, \quad B_n = 0\, \forall n \neq 2.$$

Hence the solution of the problem is

$$u(x,t) = \frac{3}{4}\sin x\cos t + \frac{1}{2}\sin 2x\sin 2t - \frac{1}{4}\sin 3x\cos 3t. \tag{1}$$

b) Since the solution contains only a finite number of smooth terms, it is verified directly that u is a classical solution of the problem.

2. It is discussed in great details in the section *5.3 Separation of variables for the wave equation* of [PR], how to solve the wave equation with Newmann conditions using the method of separation of variables, so we take the result in the book and follow the same arguments in the solution of example 5.2 in [PR]:
the solution of the problem has the form

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos n\pi x \left(A_n \cos n\pi t + B_n \sin n\pi t\right),$$

and the initial conditions are

$$\begin{aligned} u(x,0) &= \sin^2 \pi x = \frac{1}{2} - \frac{1}{2}\cos 2\pi x, \\ u_t(x,0) &= \cos \pi x, \end{aligned}$$

it follows that

$$\begin{aligned} &A_0 = 1, \quad A_2 = -\frac{1}{2}, \quad A_n = 0\, \forall n \neq 0, 2, \\ &B_1 = \frac{1}{\pi}, \quad B_n = 0\, \forall n \neq 1, \end{aligned}$$

thus the solution of the problem is

$$u(x,t) = \frac{1}{2} + \frac{1}{\pi}\cos \pi x \sin \pi t - \frac{1}{2}\cos 2\pi x \cos 2\pi t.$$

3. Consider the solution of the special form

$$u(x,t) = X(x)T(t),$$

and substitute to the equation, we can derive the equations

$$\begin{aligned} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda X &= 0, \\ \frac{\mathrm{d}T}{\mathrm{d}t} + 12\lambda T &= 0. \end{aligned}$$

The Dirichlet boundary conditions imply that for nontrivial solution u, X satisfies the boundary conditions

$$\frac{\mathrm{d}X}{\mathrm{d}x}(0) = \frac{\mathrm{d}X}{\mathrm{d}x}(\pi) = 0,$$

then the eigenvalues of the eigenvalue problem

$$\begin{aligned} &\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \lambda X = 0 \qquad\qquad 0 < x < \pi, \\ &\frac{\mathrm{d}X}{\mathrm{d}x}(0) = \frac{\mathrm{d}X}{\mathrm{d}x}(\pi) = 0 \quad t > 0, \end{aligned}$$

are

$$\lambda_n = n^2, \quad n = 0, 1, 2...$$

and the associated eigenfunctions are

$$X_n = \cos nx,$$

which are uniquely defined up to multiplicative constants. Solve the equation for T with $\lambda = \lambda_n$

$$T_n(t) = B_n e^{-12\lambda_n t} = B_n e^{-12n^2 t}, \quad n = 0, 1, 2...$$

The superposition principle implies that

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-12n^2 t} \cos nx$$

is a generalized solution of the problem. The initial condition is

$$u(x,0) = 1 + \sin^2 x = \frac{3}{2} - \frac{1}{2}\cos 2x,$$

hence the solution of the problem is

$$u(x,t) = \frac{3}{2} - \frac{1}{2}e^{-48t}\cos 2x. \tag{2}$$

4. a) Following the similar arguments solving the eigenvalue problem in p.101-102 of [PR], we can show that there are only positive eigenvalues:

$\lambda < 0$ The general solution

$$X(x) = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x} = \tilde{\alpha}\cosh\sqrt{-\lambda}x + \tilde{\beta}\sinh\sqrt{-\lambda}x,$$

thus the initial conditions imply

$$\begin{aligned} &X(0) = \tilde{\alpha}\cosh 0 + \tilde{\beta}\sinh 0 = 0, \\ &X'(\pi) = \tilde{\alpha}\sqrt{-\lambda}\sinh\sqrt{-\lambda}\pi + \tilde{\beta}\sqrt{-\lambda}\cosh\sqrt{-\lambda}\pi = 0, \end{aligned}$$

Since $\sinh 0 = 0$ and $\cosh\sqrt{-\lambda}\pi > 0$, we get $\tilde{\alpha} = \tilde{\beta} = 0$.

$\lambda = 0$ The general solution

$$X(x) = \alpha + \beta t,$$

thus the initial conditions imply

$$\begin{aligned} X(0) &= \alpha = 0, \\ X'(\pi) &= \beta = 0. \end{aligned}$$

$\lambda > 0$ The general solution

$$X(x) = \alpha \cos \sqrt{\lambda} x + \beta \sin \sqrt{\lambda} x,$$

thus the initial conditions imply

$$\begin{aligned} X(0) &= \alpha \cos 0 + \beta \sin 0 = 0 \Rightarrow \alpha = 0, \\ X'(\pi) &= -\alpha \sqrt{\lambda} \sin \sqrt{-\lambda}\pi + \beta \sqrt{\lambda} \cos \sqrt{\lambda}\pi = 0, \end{aligned}$$

thus

$$\beta \sqrt{\lambda} \cos \sqrt{\lambda}\pi = 0 \Rightarrow \lambda = (\frac{2n-1}{2})^2, \quad n = 1, 2...$$

Hence the eigenvalues of the problem are

$$\lambda_n = (\frac{2n-1}{2})^2, \quad n = 1, 2...$$

and the associated eigenfunctions are

$$X_n(x) = \beta_n \sin \frac{2n-1}{2} x, \quad n = 1, 2...$$

b) Consider the solution of special form

$$u(x, t) = X(x)T(t).$$

Substituting to the equation, we obtain the equation

$$\begin{aligned} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda X &= 0, \\ \frac{\mathrm{d}T}{\mathrm{d}t} + \lambda T &= 0. \end{aligned}$$

and the boundary conditions imply that for nontrivial solution, X has the following boundary condition

$$\begin{aligned} X(0) &= 0, \\ X'(\pi) &= 0. \end{aligned}$$

Thus by a), we have the eigenvalues

$$\lambda_n = (\frac{2n-1}{2})^2, \quad n = 1, 2...$$

and the associated eigenfunctions are

$$X_n(x) = \beta_n \sin \frac{2n-1}{2} x, \quad n = 1, 2...$$

Then we solve the equation of T with $\lambda = \lambda_n$

$$T_n(t) = e^{-\lambda_n t}$$

which is uniquely determined by a multiplicative constant. The superposition principle implies that

$$u(x, t) = \sum_{n=0}^{\infty} \beta_n e^{-(\frac{2n-1}{2})^2 t} \sin \frac{2n-1}{2} x$$

is a generalized solution of the problem. The initial condition implies that

$$u(x, t) = e^{-(\frac{3}{2})^2 t} \sin \frac{3}{2} x + e^{-(\frac{9}{2})^2 t} \sin \frac{9}{2} x = e^{-\frac{9}{4} t} \sin \frac{3}{2} x + e^{-\frac{81}{4} t} \sin \frac{9}{2} x$$

is the solution of the problem.

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).