Solution 8

1. The related eigenvalue problem is

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \lambda v = 0 \qquad 0 \leqslant x \leqslant \frac{\pi}{2},$$
$$\frac{\mathrm{d}v}{\mathrm{d}x}(0) = \frac{\mathrm{d}v}{\mathrm{d}x}(\frac{\pi}{2}) = 0.$$

The eigenvalues are given by $\lambda_n = (2n)^2$, $n = 0, 1, 2...$ and the corresponding eigenfunctions are $v_n = \cos 2nx$. We have the eigenfunction expansion of u

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos 2nx.$$

From the initial condition, we get

$$T_0(0) = 1, \quad T_2(0) = 3, \quad T_n(0) = 0 \ \forall n \neq 0, 2.$$

Substituting to the equation, we get

$$\sum_{n=0}^{\infty} \left(\frac{\mathrm{d}T_n}{\mathrm{d}t} + (2n)^2 T_n \right) \cos 2nx = 2t + 15 \cos 2x.$$

Then we get the equations for $T_n(t)$

$n = 0$

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = 2t, \quad T_0(0) = 1 \Rightarrow T_0(t) = 1 + t^2,$$

$n = 1$

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} + 4T_1 = 15, \quad T_1(0) = 0 \Rightarrow T_1(t) = \frac{15}{4} - \frac{15}{4} e^{-4t},$$

$n = 2$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} + 16T_2 = 0, \quad T_2(0) = 3 \Rightarrow T_2(t) = 3e^{-16t},$$

$n \neq 0, 1, 2$

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} + (2n)^2 T_n = 0, \quad T_n(0) = 0 \Rightarrow T_n(t) = 0.$$

Hence we get the solution of the problem

$$u(x,t) = 1 + t^2 + (\frac{15}{4} - \frac{15}{4}e^{-4t})\cos 2x + 3e^{-16t}\cos 4x.$$

2. We take the auxiliary function $w(x,t) = x \sin t$ and set $v(x,t) = u(x,t) - w(x,t)$, then $v(x,t)$ solves the following problem

$$\begin{aligned} &v_t - v_{xx} = 1 && 0 < x < 1, \quad t > 0, \\ &v_x(0,t) = v_x(1,t) = 0 && t \geqslant 0, \\ &v(x,0) = 1 + \cos(2\pi x) && 0 \leqslant x \leqslant 1. \end{aligned}$$

The related eigenvalue problem is

$$\begin{aligned} &\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \lambda v = 0 && 0 \leqslant x \leqslant 1, \\ &\frac{\mathrm{d}v}{\mathrm{d}x}(0) = \frac{\mathrm{d}v}{\mathrm{d}x}(1) = 0. \end{aligned}$$

The eigenvalues are $\lambda_n = (n\pi)^2$, $n = 0, 1, 2...$ and the corresponding eigenfunctions are $v_n = \cos n\pi x$.

v has the following eigenfunction expansion

$$v(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos n\pi x.$$

The initial conditions imply that

$$T_0(0) = 1, \quad T_2(0) = 1, \quad T_n(0) = 0 \ \forall n \neq 0, 2.$$

Substituting to the equation, we get

$$\sum_{n=0}^{\infty} \left(\frac{\mathrm{d}T_n}{\mathrm{d}t} + (n\pi)^2 T_n \right) \cos n\pi x = 1.$$

Then we get the equations for $T_n(t)$

$n = 0$

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = 1, \quad T_0(0) = 1 \Rightarrow T_0(t) = 1 + t.$$

$n = 2$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} + (2\pi)^2 T_2 = 0, \quad T_2(0) = 1 \Rightarrow T_2(t) = e^{-4\pi^2 t}.$$

$n \neq 0, 2$

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} + (n\pi)^2 T_n = 0, \quad T_n(0) = 0 \Rightarrow T_n(t) = 0.$$

Hence we get $v(x,t)$

$$v(x,t) = 1 + t + e^{-4\pi^2 t}\cos 2\pi x,$$

and the solution $u(x,t)$ of the original problem

$$u(x,t) = v(x,t) + w(x,t) = 1 + t + e^{-4\pi^2 t}\cos 2\pi x + x\sin t.$$

3. The eigenvalue problem

$$\frac{\mathrm{d}v}{\mathrm{d}x} + \lambda v = 0 \qquad 0 < x < \pi$$
$$v(0) = v(\pi) = 0$$

have eigenvalues $\lambda_n = n^2$, $n = 1, 2, 3...$ and the corresponding eigenfunctions $\lambda_n = \sin nx$. The eigenfunction expansion of $u(x,t)$ is

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin nx.$$

The initial condition implies

$$T_2(0) = 1, \quad T_n(0) = 0 \ \forall n \neq 2$$

Substituting to the equation, we have

$$\sum_{n=1}^{\infty}\left(\frac{\mathrm{d}T_n}{\mathrm{d}t} + (n^2+4)T_n\right)\sin nx = 0.$$

We get the equations for T_n:

$n = 2$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} + (2^2+4)T_2 = 0, \quad T_2(0) = 1 \Rightarrow T_2(t) = e^{-8t},$$

$n \neq 2$

$$\frac{\mathrm{d}T_n}{\mathrm{d}t} + (n^2+4)T_n = 0, \quad T_n(0) = 0 \Rightarrow T_n(t) = 0.$$

Hence $u(x,t) = e^{-8t}\sin 2x$ is the solution of the problem.

4. We take the auxiliary function $w(x,t) = \sin\frac{x}{2}$ and set $v(x,t) = u(x,t) - w(x,t)$. Then $v(x,t)$ satisfies the following equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} &v_t - v_{xx} + 4v = 0 && 0 < x < \pi, \quad t > 0, \\ &v(0,t) = 0, v(\pi,t) = 0 && t \geqslant 0, \\ &v(x,0) = \sin 2x && 0 \leqslant x \leqslant \pi. \end{aligned}$$

From the solution of ex3, we know $v(x,t) = e^{-8t}\sin 2x$ is the solution of the above problem. Hence $u(x,t) = e^{-8t}\sin 2x + \sin\frac{x}{2}$ is the solution of the original problem.

References

[PR] Y. Pinchover, J. Rubinstein, An introduction to Partial Differential Equations, Cambridge University Press(12. Mai 2005).