Exam 25.01.2017 - Solution

1.1. We apply the Laplace transform to the PDE, using the transform's properties. We get:

$$\begin{aligned}
\mathcal{L}[y'' + y](s) &= \mathcal{L}[t+2](s);\\
\mathcal{L}[y''](s) + \mathcal{L}[y](s) &= \mathcal{L}[t](s) + 2\mathcal{L}[1](s);\\
s^2\mathcal{L}[y](s) - sy(0) - y'(0) + \mathcal{L}[y](s) &= \frac{1}{s^2} + \frac{2}{s};\\
s^2\mathcal{L}[y](s) - s - 1 + \mathcal{L}[y](s) &= \frac{1}{s^2} + \frac{2}{s};\\
(s^2+1)\mathcal{L}[y](s) &= \frac{1}{s^2} + \frac{2}{s} + s + 1;\\
\mathcal{L}[y](s) &= \frac{1+2s+s^2+s^3}{s^2(s^2+1)}.
\end{aligned}$$

[2 points]

We now perform a partial fraction decomposition on the right-hand-side through the Ansatz

$$\frac{1+2s+s^2+s^3}{s^2(s^2+1)} \overset{!}{=} \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}.$$

[2 points]

We see that

$$\frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{(A+C)s^3 + (B+D)s^2 + As + B}{s^2(s^2+1)},$$

consequently the coefficients need to satisfy the system

$$\begin{cases} A + C = 1,\\ B + D = 1,\\ A = 2,\\ B = 1, \end{cases} \quad \Rightarrow \quad A = 2, \quad B = 1, \quad C = -1 \quad \text{and} \quad D = 0.$$

[2 points]

Recalling the Laplace transform of the most elementary functions we therefore deduce:

$$\begin{aligned}\mathcal{L}[y](s) &= \frac{2s+1}{s^2} - \frac{s+1}{s^2}\\ &= \frac{2}{s} + \frac{1}{s^2} - \frac{s}{s^2+1}\\ &= 2\mathcal{L}[1](s) + \mathcal{L}[t](s) - \mathcal{L}[\cos(t)](s)\\ &= \mathcal{L}[2+t-\cos(t)](s).\end{aligned}$$

[2 points]

Finally applying the inverse transformation, we conclude:

$$y(t) = 2 + t - \cos(t).$$

[2 points]

1.2. Following the hint, we first find a particular solution v of the PDE:

$$v_{tt} - 4v_{xx} = \sin(4t) + x.$$

Since on the right-hand-side space and time variables are separated, we are led to make the Ansatz $v(x,t) = v_1(x) + v_2(t)$. The PDE is then equivalent to the couple of ODEs

$$\begin{cases} -4v_1''(x) = x, \\ \quad v_2''(t) = \sin(4t), \end{cases}$$

a solution of which is readily found:

$$v_1(x) = -\frac{x^3}{24} \quad \text{and} \quad v_2(t) = -\frac{\sin(4t)}{16}.$$

We have then our particular solution:

$$v(x,t) = -\frac{x^3}{24} - \frac{\sin(4t)}{16}.$$

[3 points]

Now, if u solves the original problem, $w = u - v$ solves the homogeneous problem

$$\begin{cases} w_{tt} - 4w_{xx} = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ w(x,0) = 2x^2 + \dfrac{x^3}{24} & x \in \mathbb{R}, \\ w_t(x,0) = 6\cos(x) + \dfrac{1}{4} & x \in \mathbb{R}, \end{cases}$$

which we solve using d'Alembert's formula:

$$w(x,t) = \frac{w(x-2t,0)+w(x+2t,0)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} 6\cos(\xi) + \frac{1}{4}\,\mathrm{d}\xi$$

[3 points]

$$\begin{aligned}
&= \frac{1}{2}\left(2(x-2t)^2 + \frac{(x-2t)^3}{24} + 2(x+2t)^2 + \frac{(x+2t)^3}{24}\right) + \frac{1}{4}\left[6\sin(\xi) + \frac{\xi}{4}\right]_{x-2t}^{x+2t} \\
&= \frac{1}{2}\left(4x^2 + 16t^2 + \frac{1}{24}(2x^3 + 24xt^2)\right) \\
&\quad + \frac{1}{4}\left(6(\sin(x+2t) - \sin(x-2t)) + \frac{1}{4}(x+2t-x+2t)\right) \\
&= 2x^2 + 8t^2 + \frac{x^3}{24} + \frac{xt^2}{2} + 3\cos(x)\sin(2t) + \frac{t}{4},
\end{aligned}$$

[3 points]

where we used the trigonometric identity $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos(\alpha)\sin(\beta)$ in the last equality. We conclude that u is

$$u(x,t) = v(x,t) + w(x,t) = 2x^2 + 8t^2 + \frac{xt^2}{2} + \frac{t}{4} - \frac{\sin(4t)}{16} + 3\cos(x)\sin(2t).$$

[1 point]

Alternative solution: We use directly d'Alembert's formula for the non homogeneous wave equation:

$$\begin{aligned}
u(x,t) =& \frac{2(x-2t)^2 + 2u(x+2t)^2}{2} \\
&+ \frac{1}{4}\int_{x-2t}^{2+2t} 6\cos(\xi)\,\mathrm{d}\xi \\
&+ \frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \sin(4\tau) + \xi\,\mathrm{d}\xi\mathrm{d}\tau.
\end{aligned}$$

[3 points]

We then compute:

$$\frac{2(x-ct)^2 + 2u(x+ct)^2}{2} = 2x^2 + 8t^2,$$

[2 points]

$$\frac{1}{4}\int_{x-2t}^{2+2t} 6\cos(\xi)\,\mathrm{d}\xi = 3\cos(x)\sin(2t),$$

[2 points]

and

$$\frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \sin(4\tau) + \xi \,\mathrm{d}\xi\mathrm{d}\tau = \frac{1}{4}\left(t + 2t^2 x - \frac{1}{4}\sin(4t)\right).$$

[2 points]

We then conclude that the solution is, as before

$$u(x,t) = v(x,t) + w(x,t) = 2x^2 + 8t^2 + \frac{xt^2}{2} + \frac{t}{4} - \frac{\sin(4t)}{16} + 3\cos(x)\sin(2t).$$

[1 point]

1.3.

(a) Denote by F the 4-periodic extension of f. By construction, F is an even function, so its Fourier series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\left(\frac{\pi n x}{2}\right),$$

[1 point]

Where

$$\begin{aligned} a_n &= \frac{2}{P}\int_0^P f(x)\cos\left(\frac{2\pi n x}{P}\right)\mathrm{d}x \\ &= \frac{1}{2}\int_0^4 \frac{x^2}{2}\cos\left(\frac{\pi n x}{2}\right)\mathrm{d}x = \frac{1}{2}\int_0^2 x^2\cos\left(\frac{\pi n x}{2}\right)\mathrm{d}x. \end{aligned}$$

We now compute the a_n's. As for a_0 we get

$$a_0 = \frac{1}{2}\int_0^2 x^2\,\mathrm{d}x = \frac{4}{3}.$$

[1 point]

As for a_n, $n \geq 1$ we get, using integration by parts,

$$\begin{aligned}
a_n &= \frac{1}{2}\int_0^2 x^2 \cos\left(\frac{\pi n x}{2}\right) \mathrm{d}x \\
&= \frac{1}{2}\left(\underbrace{\frac{x^2 \sin(\pi n x/2)}{\pi n/2}\bigg|_0^2}_{=0} - \int_0^2 \frac{2x\sin(\pi n x/2)}{\pi n/2}\,\mathrm{d}x \right) \\
&= \frac{1}{2}\left(\frac{2x\cos(\pi n x/2)}{(\pi n/2)^2}\bigg|_0^2 - \int_0^2 \frac{2\cos(\pi n x/2)}{(\pi n/2)^2}\,\mathrm{d}x \right) \\
&= \frac{1}{2}\left((-1)^n \frac{16}{\pi^2 n^2} - \underbrace{\frac{2\sin(\pi n x/2)}{(\pi n/2)^3}\bigg|_0^2}_{=0} \right) = (-1)^n \frac{8}{\pi^2 n^2}.
\end{aligned}$$

[3 points]

We conclude

$$F(x) \sim \frac{2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{8}{\pi^2 n^2} \cos\left(\frac{\pi n x}{2}\right).$$

[1 point]

(b) *Since F is continuous and piecewise differentiable, its Fourier series converges uniformly, and we can thus exchange "$\sim$" with "$=$".* *[1 point]*

In order to compute the requested series, we see that it is convenient to evaluate in 0 F, so that

$$0 = F(0) = \frac{2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{8}{\pi^2 n^2} \quad \Rightarrow \quad -\frac{\pi^2}{12} = \sum_{n=1}^{+\infty} (-1)^n \frac{1}{n^2}.$$

Multiplying both hand-sides by -1 leads to the result:

$$\frac{\pi^2}{12} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n^2}.$$

[3 points]

1.4. Let us recall that for $a > 0$ there holds

$$\mathcal{F}[e^{-ax^2}](\xi) = \int_{\mathbb{R}} e^{-ax^2} \mathrm{e}^{i\xi x}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\mathrm{e}^{-\frac{\xi^2}{4a}}.$$

So in our case where $a = 2$ we get

$$\mathcal{F}[e^{-2x^2}](\xi) = \sqrt{\frac{\pi}{2}}\mathrm{e}^{-\frac{\xi^2}{8}}.$$

[2 points]

From the well-known relation between derivatives and Fourier transform:

$$\frac{\mathrm{d}^k \mathcal{F}[f](\xi)}{\mathrm{d}\xi^2} = (-i)^k \mathcal{F}[x^k f(x)](\xi),$$

we deduce, in our case where $k = 2$ that

$$\int_{\mathbb{R}} x^2 f(x)\,\mathrm{d}x = -\frac{\mathrm{d}^k \mathcal{F}[f](0)}{\mathrm{d}\xi^2},$$

[4 points]

Now, there holds

$$\frac{\mathrm{d}\mathcal{F}[f](\xi)}{\mathrm{d}\xi} = -\sqrt{\frac{\pi}{2}}\frac{2\xi}{8}\mathrm{e}^{-\frac{\xi^2}{8}},$$
$$\frac{\mathrm{d}^2\mathcal{F}[f](\xi)}{\mathrm{d}^2\xi} = \sqrt{\frac{\pi}{2}}\left(-\frac{2}{8}\mathrm{e}^{-\frac{\xi^2}{8}} + \left(\frac{2\xi}{8}\right)^2 \mathrm{e}^{-\frac{\xi^2}{8}}\right),$$

[2 points]

We conclude that

$$\int_{\mathbb{R}} x^2 f(x)\,\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{2}}.$$

[2 points]

1.5.

(a) From the Maximum principle, we know that u attains its maximum and minimum on the boundary. *[1 point]*

Being there $u(x, y) = 1 + x^2$, we clearly see that the minimum value is attained at the points where x^2 is minimum, for example $(x, y) = (0, 2)$. We then see that $u(0, 2) = 1$. An identical strategy applies for the maximum: it is attained where x^2 is maximum, for example, at the point $(x, y) = (2, 0)$, where we see that $u(2, 0) = 5$.

We conclude that $1 \le u \le 5$ on $B_2(0)$. *[2 points]*

(b) Following the hint we switch to polar coordinates:

$$\begin{cases} x = r\cos(\theta), \\ y = r\sin(\theta), \end{cases} \quad \text{for } (r,\theta) \in (0,2) \times (0,2\pi).$$

As we know from the method of separation of variables, if we expand the boundary datum in Fourier series

$$u(2,\theta) \to \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(n\theta) + b_n \sin(n\theta),$$

then $u(r,\theta)$ will be given by

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(\frac{r}{2}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

[3 points]

Now, the boundary datum takes the form:

$$u(2,\theta) = 1 + 4\cos^2(\theta),$$

which, thanks to the trigonometric identity

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2},$$

we can rewrite as $u(2,\theta) = 3 + 2\cos(2\theta)$. *[2 points]*

We conclude that

$$u(r,\theta) = 3 + \frac{r^2}{2}\cos(2\theta).$$

[2 points]

We remark that, reverting to Cartesian coordinates, the above expression becomes

$$u(x,y) = 3 + \frac{x^2 - y^2}{2}.$$