NumCSE

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Exercise sheet 8
Splines

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Problem 8.1: Cubic splines

We implement interpolation of a discrete data set by a cubic spline.

Template: `CubicSplines.cpp`

Recall that the cubic spline s interpolating a given data set $(t_0, y_0), \ldots, (t_n, y_n)$ is a C^2 function on $[t_0, t_n]$ which is a polynomial of third degree on every subinterval $[t_j, t_{j+1}]$ for $j = 0, \ldots, n-1$, and such that $s(t_j) = y_j$ for every $j = 0, \ldots, n$. To ensure uniquenss we impose the additional boundary conditions $s''(t_0) = s''(t_n) = 0$.

Recall that since we can represent a polynomial of degree d as a vector of length $d+1$ which contains the polynomial's coefficients, a cubic spline on a data set of length $n+1$ can be represented as a $4 \times n$ matrix, where the column j specifies the coefficients of the interpolating polynomial on the interval $[t_j, t_j + 1]$.

(a) Implement a `C++` function `cubicSpline` which takes as input vectors T and Y, and returns the matrix representing the cubic spline which interpolates them.

Hint: implement the formulae from the tablet notes to calculate the second derivatives of the splines in the points t_j, then use them to build the matrix associated to the spline.

(b) Implement a `C++` function which given a cubic spline, its interpolation nodes and a vector of evaluation points, returns the value the spline takes on the evaluation points.

(c) Run some tests of your spline evaluation function (see template).

Problem 8.2: Piecewise linear approximation on graded meshes

The quality of an interpolation depends heavily on the choice of the nodes: for instance if the function to be interpolated has very large derivatives on a part of the domain, more interpolation points will be required there. Commonly used tools to cope with this task are *graded meshes*, which are explored in this problem.

Given a mesh $\mathcal{T} = \{0 \le t_0 < t_1 < \cdots < t_n \le 1\}$ on the unit interval $I = [0, 1]$, we define the *piecewise linear interpolant*:

$$\mathsf{I}_{\mathcal{T}} : C^0(I) \to \mathcal{P}_{1,\mathcal{T}} = \{s \in C^0(I),\ s_{|[t_{j-1},t_j]} \in \mathcal{P}_1\ \forall\, j\}, \quad \text{s.t.} \quad \big(\mathsf{I}_{\mathcal{T}} f\big)(t_j) = f(t_j), \quad j = 0, \ldots, n.$$

(a) If we choose the uniform mesh $\mathcal{T} = \{t_j\}_{j=0}^{n}$ with $t_j = j/n$, given a function $f \in C^2(I)$ what is the asymptotic behavior of the error $\max_{x \in I} |f(x) - \mathsf{I}_{\mathcal{T}} f(x)|$ when $n \to \infty$?

Hint: use the following property of the interpolating polynomial: for every j, there exists $\xi_j \in [t_j, t_{j+1}]$ such that

$$f(t) - p_j(t) = \frac{f''(\xi_j)}{6}(t - t_j)(t - t_{j+1}), \qquad \text{for } t \in [t_j, t_{j+1}],$$

where p_j is the linear interpolant of f in $[t_j, t_{j+1}]$.

(b) What is the regularity of the function $f : I \to \mathbb{R},\ f(t) = t^{\alpha},\ 0 < \alpha < 2$? In other words, for which $k \in \mathbb{N}$ do we have $f \in C^k(I)$?

Hint: check the continuity of the derivatives in the endpoints of I.

(c) Study with some numerical experiments the convergence of the piecewise linear approximation of $f(t) = t^{\alpha}$ (with $0 < \alpha < 2$) on uniform meshes.

(d) In which mesh interval do you expect $|f - \mathsf{I}_{\mathcal{T}} f|$ to attain its maximum?

(e) Compute by hand the exact value of $\|f - \mathsf{I}_{\mathcal{T}} f\|_{L^\infty(I)}$. Use the result of the Point (d) to simplify the problem. Compare the order of convergence obtained with the one observed numerically.

(f) Since the interpolation error is concentrated in the left part of the domain, it seems reasonable to use a finer mesh only in this part. A common choice is an *algebraically graded mesh*, defined as $\mathcal{G} = \left\{ t_j = \left(\frac{j}{n}\right)^{\beta}, \quad j = 0, \ldots, n \right\}$ for a parameter $\beta > 1$. An example is depicted in Fig. 1 for $\beta = 2$.

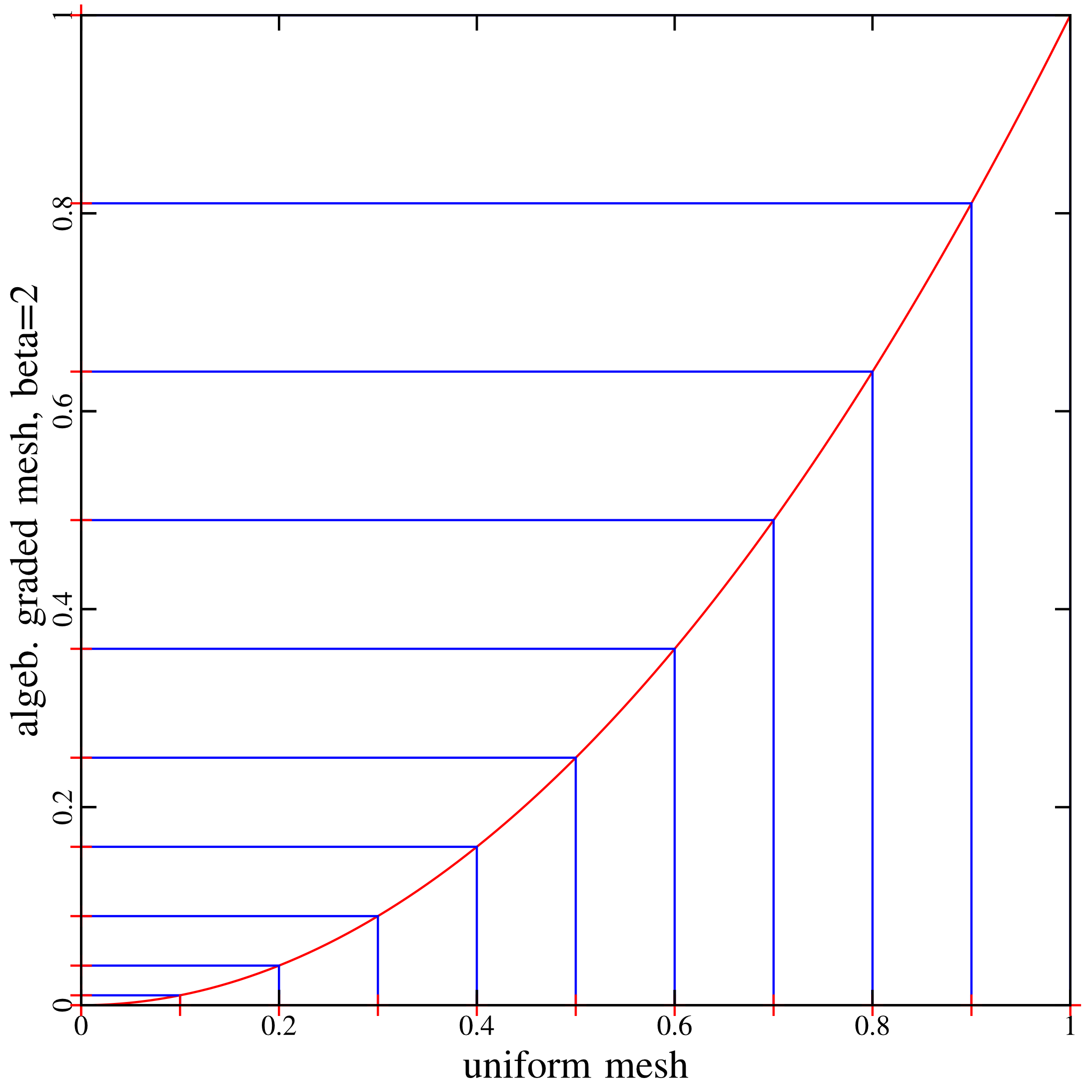

Fig. 1

For a fixed parameter α in the definition of f, determine with a numerical experiment the rate of convergence of the piecewise linear interpolant $\mathsf{I}_{\mathcal{G}}$ on the graded mesh $\mathcal{G}$ as a function of the parameter β. Try for instance $\alpha = 1/2$, $\alpha = 3/4$ or $\alpha = 4/3$.

How do you have to choose β in order to recover the optimal rate $\mathcal{O}(n^{-2})$ (if possible)?