Answer sheet compilation instructions

- Use only black or blue pen.

For open answers:

- Write clearly only **inside** boxes, away from the borders.
- Write a **single** character (number or letter) per box.
- Start writing from the left, leaving empty boxes on the right.

For multiple choice and true/false questions:

- **Fill** the circle for the answer you consider correct (only one answer is correct).
- Remarks and computations have **no** influence on points awarded.
- Any unclear or double marking will be considered as an answer not given (0 points).
- Wrong answers give **negative** points.

Exam instructions

- Turn off your devices and leave them in your bag.
- Only pens and Legi should be on the table.
- Fill last name and Legi number on the answer sheet.
- **Turn this sheet only when instructed to do so.**
- At the end of the exam, take everything except the single answer sheet which you want to submit.

Questions

NumCSE midterm, HS 2017

1. *Cancellation error* [12 P.]

   Which of the following expressions can be affected by cancellation errors for some choice of x in the specified interval? (True = affected by cancellation, False = **not** affected by cancellation).

   (a) $y = x^2 - \sqrt{x^2 + 2}$, for $x \in [2, \infty)$. F

   (b) $y = \dfrac{1 - \cos(x)}{x^2}$, for $x \in (0, \dfrac{\pi}{2})$. T

   (c) $y = \log_2(x - \sqrt{x - 1})$, for $x \in [1, \infty)$. F

   (d) $y = \exp(x) - \exp(2(x - 1))$, for $x \in (1, 2]$. T

2. *Numerical Stability* [8 P.]

   Let $\tilde{\mathbf{F}} : \mathbf{X} \mapsto \tilde{\mathbf{Y}}$ be an algorithm for the problem $\mathbf{F} : \mathbf{X} \mapsto \mathbf{Y}$.

   (a) True or false: if $\tilde{\mathbf{F}}$ is backward stable then, for any $\mathbf{x} \in \mathbf{X}$, $\tilde{\mathbf{F}}(\mathbf{x})$ will be close to $\mathbf{F}(\mathbf{x})$. False

   (b) True or false: backward stability of $\tilde{\mathbf{F}}$ implies mixed stability. True

   (c) Among the following, choose the best definition of condition number of $\mathbf{F}$ in $\mathbf{x}$:

   (i) $\sup\limits_{\Delta x \text{ small}} \left(\frac{||\mathbf{F}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{F}(\mathbf{x})||}{||\mathbf{F}(\mathbf{x})||} \cdot \frac{||\Delta\mathbf{x}||}{||\mathbf{x}||} \right)$

   (ii) $\sup\limits_{\Delta x \text{ small}} \left(\frac{||\mathbf{F}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{F}(\mathbf{x})||}{||\mathbf{F}(\mathbf{x})||} \cdot \frac{||\mathbf{x}||}{||\Delta\mathbf{x}||} \right)$

   (iii) $\inf\limits_{\Delta x \text{ small}} \left(\frac{||\mathbf{F}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{F}(\mathbf{x})||}{||\mathbf{F}(\mathbf{x})||} \cdot \frac{||\Delta\mathbf{x}||}{||\mathbf{x}||} \right)$

   (iv) $\inf\limits_{\Delta x \text{ small}} \left(\frac{||\mathbf{F}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{F}(\mathbf{x})||}{||\mathbf{F}(\mathbf{x})||} \cdot \frac{||\mathbf{x}||}{||\Delta\mathbf{x}||} \right)$

3. *Singular Value Decomposition* [9 P.]

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 36 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{U}, \mathbf{\Sigma}, \mathbf{V}$ be the matrices involved in the **thin** singular value decomposition of $\mathbf{A}$ (in particular $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$).

(a) What is the condition number of $\mathbf{\Sigma}$? $36/3 = 12$ or ∞

(b) What is the condition number of $\mathbf{\Sigma}^\top\mathbf{\Sigma}$? $(12)^2 = 144$ or ∞

(c) What are the dimensions of $\mathbf{U}$ (number of rows $\times$ number of columns)?

5×3 or 5x5

4. *Linear system solution through LU decomposition* [9 P.]

Consider the following `C++/Eigen` function:

```
1  int solve_triang(int n) {
2      using namespace Eigen;
3      MatrixXd A = MatrixXd::Zero(n,n);
4
5      for (int i=0; i < n; i++) {
6          for (int j=i; j < n; j++) {
7              // fill upper triangular part of A
8              A(i,j) = 2;
9          }
10     }
11
12     // fill b
13     VectorXd b = 46 * VectorXd::Ones(n);
14
15     // compute solution to Ax = b
16     VectorXd x = A.fullPivLu().solve(b);
17
18     return x(n-1);
19 }
```

(a) What is the value returned by `solve_triang(2017)` ? 23

(b) Choose the lowest correct asymptotic complexity of the function as $n \to \infty$:

(i) $O(n \log n)$
(ii) $O(n^2)$
(iii) $O(n^2 \log n)$
(iv) $O(n^3)$

(c) Suppose we replace line 16 with

`VectorXd x = A.triangularView<Upper>().solve(b);`

Choose the lowest correct asymptotic complexity of the modified function as $n \to \infty$:

(i) $O(n \log n)$
(ii) $O(n^2)$
(iii) $O(n^2 \log n)$
(iv) $O(n^3)$

5. *Linear Least Squares* [12 P.]

Consider the linear least squares problem:

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m,n}$ is a large sparse matrix and $\mathbf{b} \in \mathbb{R}^m$, for $m \geq n$ and $m, n \in \mathbb{N}$.

(a) State whether the following relations are true or false:

(i) $\mathcal{N}(\mathbf{A}^\top\mathbf{A}) = \mathcal{R}(\mathbf{A}^\top)^\perp$ T

(ii) $\mathcal{N}(\mathbf{A}^\top\mathbf{A}) = \mathcal{N}(\mathbf{A})^\perp$ F

(iii) $\mathcal{R}(\mathbf{A}^\top\mathbf{A}) = \mathcal{R}(\mathbf{A}^\top)$ T

(iv) $\mathcal{R}(\mathbf{A}^\top\mathbf{A}) = \mathcal{N}(\mathbf{A})^\perp$ T

Here $\mathcal{R}$ stands for range and $\mathcal{N}$ for null-space.

(b) If $\mathbf{x}^*$ is unique, then what is the rank of $\mathbf{A}$? n

(c) Assuming $\mathbf{A}$ is well-conditioned and a unique solution exists, choose the most efficient method to solve (1) among the following:

(i) Normal equations

(ii) Householder QR decomposition

~~(iii)~~ Extended normal equations

(iv) Singular value decomposition