Numerical Methods for Computational Science and Engineering

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Prof. Rima Alaifari, SAM, ETH Zurich

Recap: Theorem 6.1.6: for every $f \in C^0([0,1])$

sequence of Bernstein approximants p_n, s.t.

$\|p_n - f\|_\infty \to 0$ for $n \to \infty$

But: slow convergence

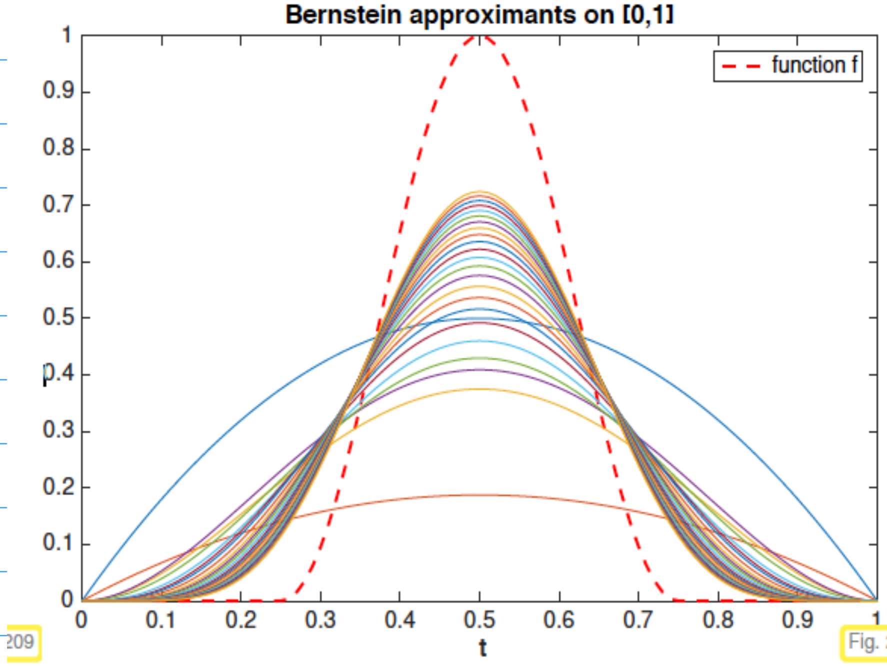
See proof in lecture notes:

$\forall\, t \in [0,1]: \; |f(t) - p_n(t)| \leq (\|f\|_\infty + 1)\,\varepsilon$

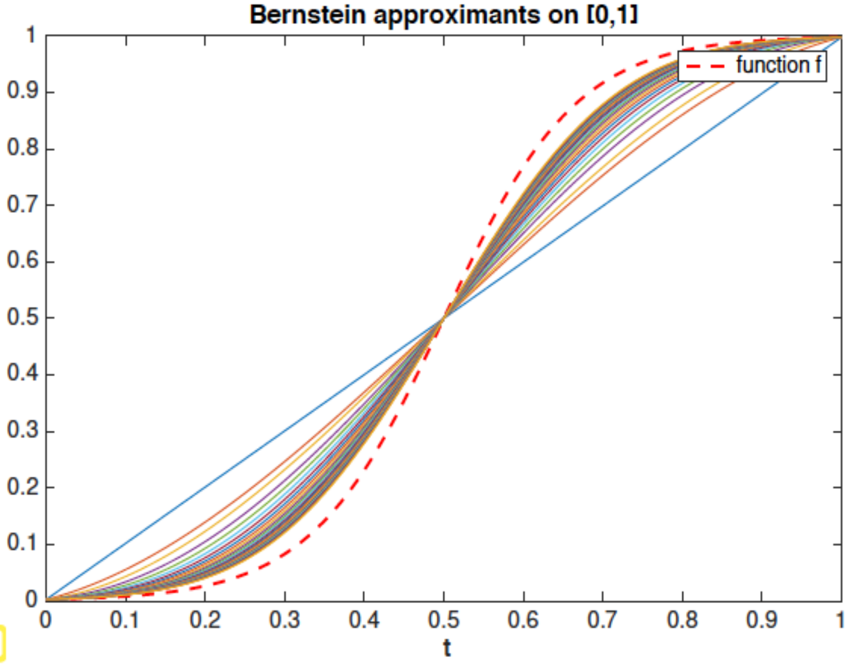
$\varepsilon \lesssim \frac{1}{n^\alpha} \qquad \alpha < 1$

Examples:

$$f_1(t) := \begin{cases} 0 & \text{, if } |2t-1| > \frac{1}{2}, \\ \frac{1}{2}(1+\cos(2\pi(2t-1))) & \text{else,} \end{cases} \quad , \quad f_2(t) := \frac{1}{1+e^{-12(x-1/2)}} .$$



$f = f_1$



$f = f_2$

$p_n \qquad n = 1/2, \dots, 25$

Are there polynomials that converge faster?

Requires notion of best approximation error

Definition 6.1.14. (Size of) best approximaton error

Let $\|\cdot\|$ be a (semi-)norm on a space X of functions $I \mapsto \mathbb{K}$, $I \subset \mathbb{R}$ an interval. The (size of the) best approximation error of $f \in X$ in the space $\mathcal{P}_k$ of polynomials of degree $\leq k$ with respect to $\|\cdot\|$ is

$$\operatorname{dist}_{\|\cdot\|}(f, \mathcal{P}_k) := \inf_{p \in \mathcal{P}_k} \|f - p\| .$$

Best possible L^∞ approximation:

Theorem 6.1.15. L^∞ polynomial best approximation estimate [Jackson's thm]

If $f \in C^r([-1,1])$ (r times continuously differentiable), $r \in \mathbb{N}$, then, for any polynomial degree $n \geq r$,

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([-1,1])} \leq (1 + \pi^2/2)^r \boxed{\frac{(n-r)!}{n!}} \left\|f^{(r)}\right\|_{L^\infty([-1,1])} .$$

$\inf_{p \in \mathcal{P}_n} \|f-p\|_{L^\infty([-1,1])}$

- norm of best appr. error
- always exists ($\mathcal{P}_n$ is finite-dim)
- uniform approximation

Note: This estimate depends on smoothness of f!

$r=1$: $\inf_{p \in \mathcal{P}_n} \|f-p\|_{L^\infty([-1,1])} \leq C \cdot \frac{1}{n} \cdot \|f'\|_{L^\infty([-1,1])}$

$r=2$: $\inf_{p \in \mathcal{P}_n} \|f-p\|_{L^\infty([-1,1])} \leq C^2 \frac{1}{n(n-1)} \|f''\|_{L^\infty([-1,1])}$

Using Sterling's formula:

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n} \leq n! \leq e^{1-n} n^{n+1/2}$$

$$\frac{(n-r)!}{n!} \leq \frac{e^{1-(n-r)} (n-r)^{n-r+1/2}}{\sqrt{2\pi}\, n^{n+1/2} e^{-n}}$$

$$\leq e^{1+r} \frac{1}{\sqrt{2\pi}} \frac{(n-r)^{n-r+1/2}}{n^{n+1/2-r} \cdot n^r}$$

$\frac{n-r}{n} \leq 1$, $n-r+\frac{1}{2} > 0$ $\rightarrow$ $\leq \tilde{C}(r) \cdot n^{-r}$

$$\inf_{p\in\mathbb{P}_n} \|f-p\|_{L^\infty([-1,1])} \le C(r)\, n^{-r}\, \|f^{(r)}\|_{L^\infty([-1,1])}$$

for some constant $C(r)$ depending on r but independent of f and n.

In asymptotic form:

$$\inf_{p\in\mathbb{P}_n} \|f-p\|_{L^\infty([-1,1])} = \mathcal{O}(n^{-r}) \quad \text{as } n\to\infty$$

algebraic convergence in n

governed by smoothness of f!

What if $f\in C^\infty$?

$\forall\, r\in\mathbb{N}$: $\exists\; C_1 = C_1(r)$ s.t.

$$\inf \|f-p\|_\infty \le C_1(r)\, n^{-r}$$

$$C_1(r) = C(r)\, \|f^{(r)}\|_\infty = \left(1+\frac{\pi^2}{2}\right) \tilde{C}(r)\, \|f^{(r)}\|_\infty$$

Note: $\|f^{(r)}\|_\infty$ might not be uniformly bounded in r

Also: Possibly $\tilde{C}(r)\to\infty$ as $r\to\infty$

(for example: $r=\frac{n}{2}$

$$\tilde{C}(r) = e^{1+n/2}\left(\frac{1}{2}\right)^{\frac{n+1}{2}} \to \infty \quad \text{as } n\to\infty)$$

Polynomial approximations on arbitrary intervals $[a,b]$:

Approximation schemes on $[0,1]$ or $[-1,1]$ can be transformed to appr. scheme on $[a,b]$?

Idea: use affine linear mapping

$$\Phi: [-1,1] \to [a,b]$$

$$\Phi(\hat{t}) = a + \frac{1}{2}(\hat{t}+1)(b-a) \qquad \hat{t}\in[-1,1]$$

to introduce affine linear pullback of f:

$$\Phi^* : C^0([a,b]) \to C^0([-1,1]) \quad , \quad \Phi^*(f)(\hat{t}) := f(\Phi(\hat{t}))\,, \quad -1 \le \hat{t} \le 1 \,. \tag{6.1.20}$$

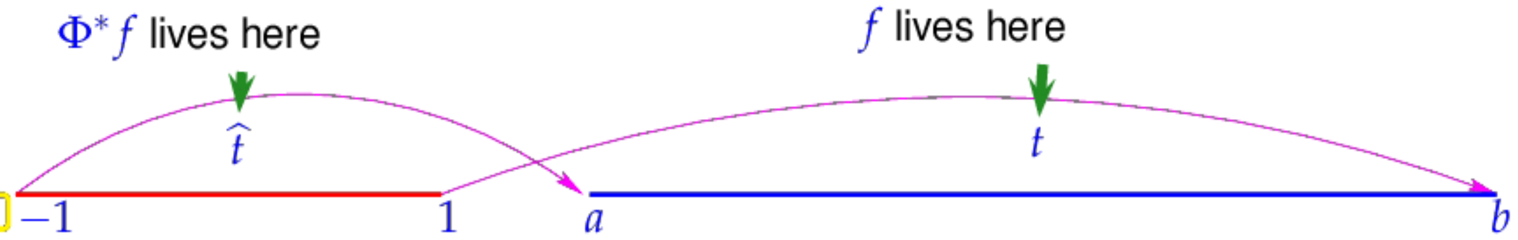


Fig. 211

$$\hat{t} \mapsto t := \Phi(\hat{t}) := \tfrac{1}{2}(1-\hat{t})a + \tfrac{1}{2}(\hat{t}+1)b$$

$$\Phi^* f = f \circ \Phi \in C^0([-1,1])$$

Lemma 6.1.21. Affine pullbacks preserve polynomials

If $\Phi^* : C^0([a,b]) \to C^0([-1,1])$ is an affine pullback according to (6.1.19) and (6.1.20), then $\Phi^* : \mathcal{P}_n \to \mathcal{P}_n$ is a bijective linear mapping for any $n \in \mathbb{N}_0$.

$\Rightarrow$ For $\hat{A}$ an appr. scheme on $C^0([-1,1])$

we can define an appr. scheme A on $C^0([a,b])$:

$A : C^0([a,b]) \to \mathcal{P}_n$

$\hat{A} : C^0([-1,1]) \to \mathcal{P}_n$

$A := (\Phi^*)^{-1} \circ \hat{A} \circ \Phi^*$ $\qquad \Phi^* : C^0([a,b]) \to C^0([-1,1])$

What is $(\Phi^*)^{-1}$?

$f = (\Phi^*)^{-1} \underbrace{\Phi^* f}_{f \circ \Phi}$ $\qquad g = \Phi^* (\Phi^*)^{-1} g$

$(\Phi^*)^{-1} g = g \circ \Phi^{-1}$ $\qquad f = f \circ \Phi \circ \Phi^{-1}$

$\qquad g = g \circ \Phi^{-1} \circ \Phi$

$\Phi^{-1} : [a,b] \to [-1,1]$

$\qquad t \mapsto \frac{2}{b-a} t - \frac{a+b}{b-a}$

ad Lemma 6.1.21

pullback is linear $\rightsquigarrow$ suffices to check its action on monomials:

$$\Phi^*\{t \mapsto t^n\} = \{\hat{t} \mapsto (a + \tfrac{1}{2}(\hat{t}+1)(b-a))^n\} \in \mathcal{P}_n$$

$$\Phi^* : \mathcal{P}_n \to \mathcal{P}_n$$

and restriction of $(\Phi^*)^{-1}$ to $\mathcal{P}_n$ has the same property:

$$(\Phi^*)^{-1} : \mathcal{P}_n \to \mathcal{P}_n \quad .$$

Transformation of norms:

Lemma 6.1.24. Transformation of norms under affine pullbacks

For every $f \in C^0([a,b])$ *we have*

$$\|f\|_{L^\infty([a,b])} = \|\Phi^* f\|_{L^\infty([-1,1])} \quad , \quad \|f\|_{L^2([a,b])} = \sqrt{\frac{|b-a|}{2}}\,\|\Phi^* f\|_{L^2([-1,1])} \; . \qquad (6.1.25)$$

clear because Φ^* doesn't change point values

$$\|f\|^2_{L^2([a,b])} = \int_a^b |f(t)|^2\,dt = \int_{-1}^{1} |f(\Phi(\hat{t}))|^2\,\Phi'(\hat{t})\,d\hat{t}$$

$$= \frac{b-a}{2}\int_{-1}^{1} |\Phi^* f(\hat{t})|^2\,d\hat{t}$$

$$= \frac{b-a}{2}\,\|\Phi^* f\|^2_{L^2([-1,1])}$$

If A is approximation scheme for $f \in C^0([a,b])$:

$$\|f - Af\|_{L^\infty([a,b])} = \|\Phi^* f - \hat{A}(\Phi^* f)\|_{L^\infty([-1,1])}$$

$$\|f - Af\|_{L^2([a,b])} = \sqrt{\frac{|b-a|}{2}}\,\|\Phi^* f - \hat{A}(\Phi^* f)\|_{L^2([-1,1])}$$

What can we say about $\|f^{(r)}\|_{L^\infty([a,b])}$?

Employ chain rule:

$$(\Phi^* f)'(\hat{t}) = f'(\Phi(\hat{t})) \cdot \underbrace{\Phi'(\hat{t})}_{\frac{1}{2}(b-a)}$$

$$\Rightarrow \quad (\Phi^* f)^{(r)}(\hat{t}) = \left(\frac{b-a}{2}\right)^r \Phi^*(f^{(r)})(\hat{t})$$

$$\Rightarrow \quad \|\Phi^* f^{(r)}\|_{L^\infty([-1,1])} = \left(\frac{b-a}{2}\right)^r \|f^{(r)}\|_{L^\infty([a,b])} \qquad (*)$$

Best polynomial approximation on $C^0([a,b])$?

[Jackson's thm for arbitrary intervals]

$$\inf_{p\in\mathcal{P}_n} \|f-p\|_{L^\infty([a,b])} = \inf_{p\in\mathcal{P}_n} \|\Phi^* f - \Phi^* p\|_{L^\infty([-1,1])}$$

$$\underset{\Phi^*(\mathcal{P}_n)=\mathcal{P}_n}{=} \inf_{p\in\mathcal{P}_n} \|\Phi^* f - p\|_{L^\infty([-1,1])}$$

$$\leq \left(1+\frac{\pi^2}{2}\right)^r \frac{(n-r)!}{n!} \|(\Phi^* f)^{(r)}\|_{L^\infty([-1,1])}$$

$$\underset{(*)}{\leq} \left(1+\frac{\pi^2}{2}\right)^r \frac{(n-r)!}{n!} \left(\frac{b-a}{2}\right)^r \|f^{(r)}\|_{L^\infty([a,b])}$$

$$\Rightarrow \quad \inf_{p\in\mathcal{P}_n} \|f-p\|_{L^\infty([a,b])} \underset{\text{Stirling}}{\leq} C(r)\left(\frac{b-a}{2n}\right)^r \|f^{(r)}\|_{L^\infty([a,b])}$$

algebraic convergence, but constant also depends on $|I| = |b-a|$.

6.1.2. Error estimates for polynomial interpolation

Definition 6.1.32. Lagrangian (interpolation polynomial) approximation scheme

Given an interval $I \subset \mathbb{R}$, $n \in \mathbb{N}$, a node set $\mathcal{T} = \{t_0, \dots, t_n\} \subset I$, the Lagrangian (interpolation polynomial) approximation scheme $\mathsf{L}_{\mathcal{T}} : C^0(I) \to \mathcal{P}_n$ is defined by

$$\mathsf{L}_{\mathcal{T}}(f) := \mathsf{I}_{\mathcal{T}}(\mathbf{y}) \in \mathcal{P}_n \quad \text{with} \quad \mathbf{y} := (f(t_0), \dots, f(t_n))^T \in \mathbb{K}^{n+1}.$$

Behavior as number of nodes is increased:

Different families $\mathcal{T}_n = \{t_0^{(n)}, \dots, t_n^{(n)}\}$ of nodes

$$\mathcal{T}_n := \{ t_j^{(n)} := a + (b-a)\frac{j}{n},\ j = 0, \dots, n \} \subset I$$

For family of polynomial approximation schemes $\{A_n\}_{n \in \mathbb{N}}$:

$$\|f - \underbrace{A_n f}\| \overset{?}{\leq} T(n) \quad \text{for } n \to \infty$$

here: Lagrange interp. with equidistant nodes

i.e. $\|f - \mathsf{L}_{\mathcal{T}_n} f\| \overset{?}{\leq} T(n)$ [bound on interp. error]

Example: $f(t) = \sin(t)$ $\quad I = [0, \pi]$

Lagrange interp. with equidistant nodes

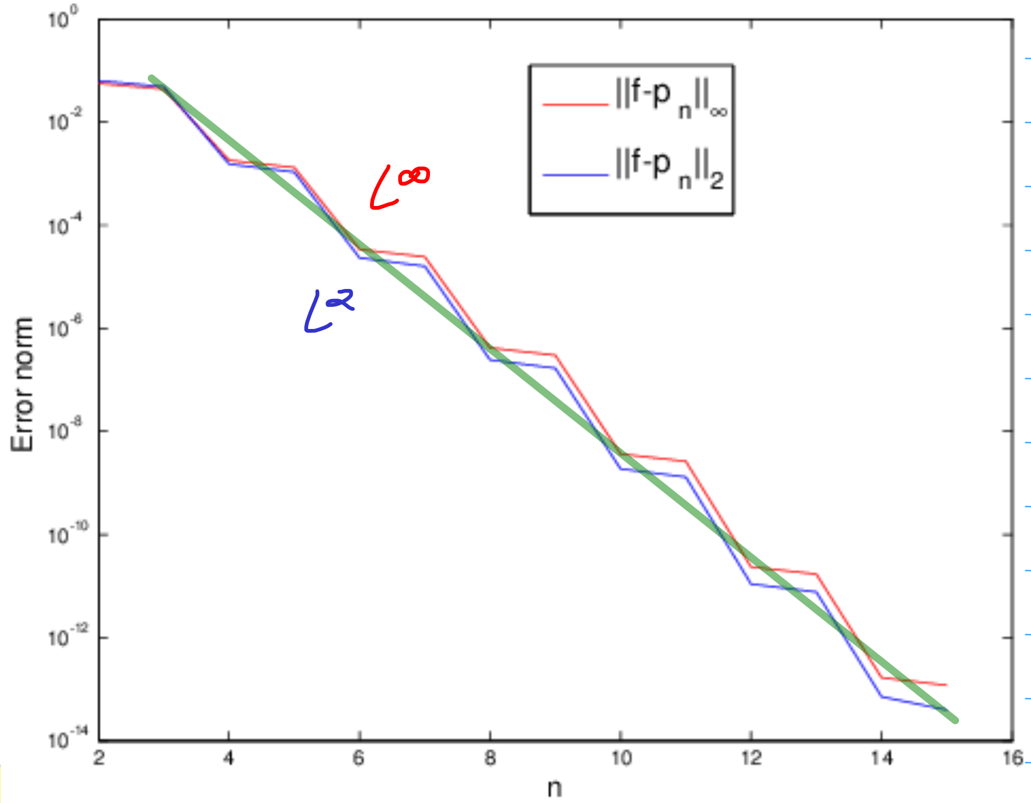


Fig. 212

lin-log plot

$\varepsilon_n := \|f - \mathsf{L}_{\mathcal{T}_n} f\|$ observation: $\log \varepsilon_n \approx C - Kn$, $K > 0$

roughly linear

$\varepsilon_n \approx e^{C} e^{-kn}$

i.e. $\varepsilon = \Theta(q^n)$ for some $q \in (0,1)$, $n \to \infty$

exponential convergence

Algebraic convergence: $\|f - \mathsf{I}_{\mathcal{T}} f\| = O(n^{-p})$
Exponential convergence: $\|f - \mathsf{I}_{\mathcal{T}} f\| = O(q^n)$
for $n \to \infty$ ("asymptotic!")

In this example: $\sin(t) \in C^\infty$

simple Lagrange interpolation is doing much better than predicted by Jackson's thm [which is always for fixed smoothness r]

Example: $f(t) = \dfrac{1}{1+t^2}$ $t \in \mathbb{R}$ $I = [-5,5]$

again equidistant Lagrange interpolation

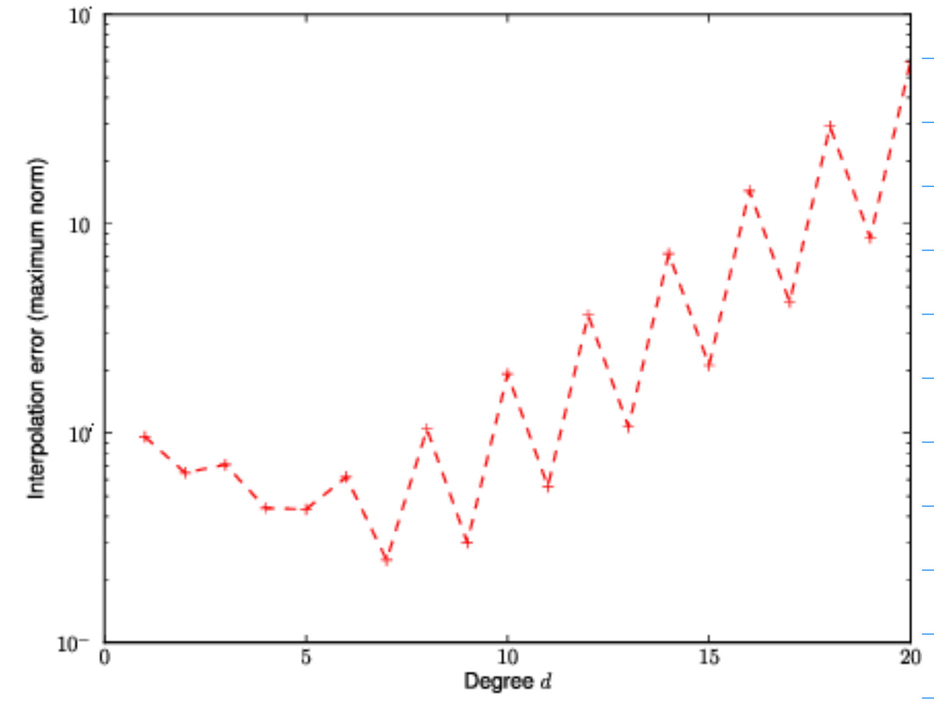


Fig. 214

Approximate $\|f - \mathsf{L}_{\mathcal{T}_n} f\|_\infty$ on $[-5,5]$

Recall:

Runge's phenomenon

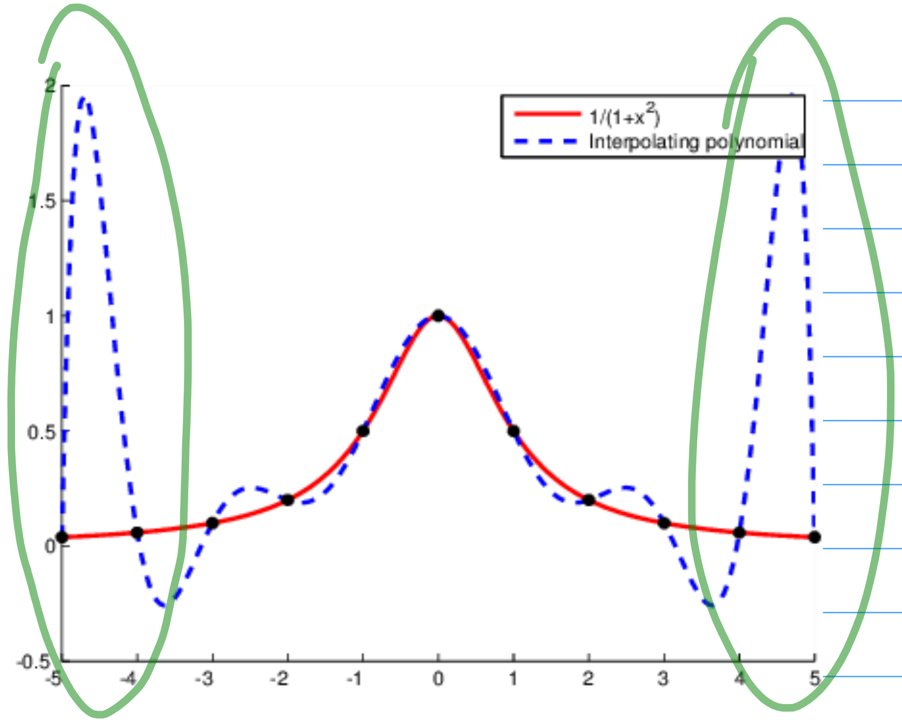


Fig. 213

Interpolating polynomial, $n = 10$

Can we understand the interpolation error?

Theorem 6.1.44. Representation of interpolation error [?, Thm. 8.22], [?, Thm. 37.4]

We consider $f \in C^{n+1}(I)$ and the Lagrangian interpolation approximation scheme ($\to$ Def. 6.1.32) for a node set $\mathcal{T} := \{t_0, \ldots, t_n\} \subset I$. Then,

for every $t \in I$ there exists a $\tau_t \in]\min\{t, t_0, \ldots, t_n\}, \max\{t, t_0, \ldots, t_n\}[$ such that

$$f(t) - \mathsf{L}_{\mathcal{T}}(f)(t) = \underbrace{\frac{f^{(n+1)}(\tau_t)}{(n+1)!}}_{=c} \cdot \underbrace{\prod_{j=0}^{n}(t - t_j)}_{=:\,\omega(t)} . \tag{6.1.45}$$

"nodal polynomial"

Proof: Fix $t \in I \setminus \mathcal{T}$ $\qquad f \in C^{n+1}(I)$

We can choose $c \in \mathbb{R}$ s.t.

$$\underset{\uparrow \text{ fixed}}{f(t)} - L_{\mathcal{T}} f(t) - c\,\omega(t) = 0$$

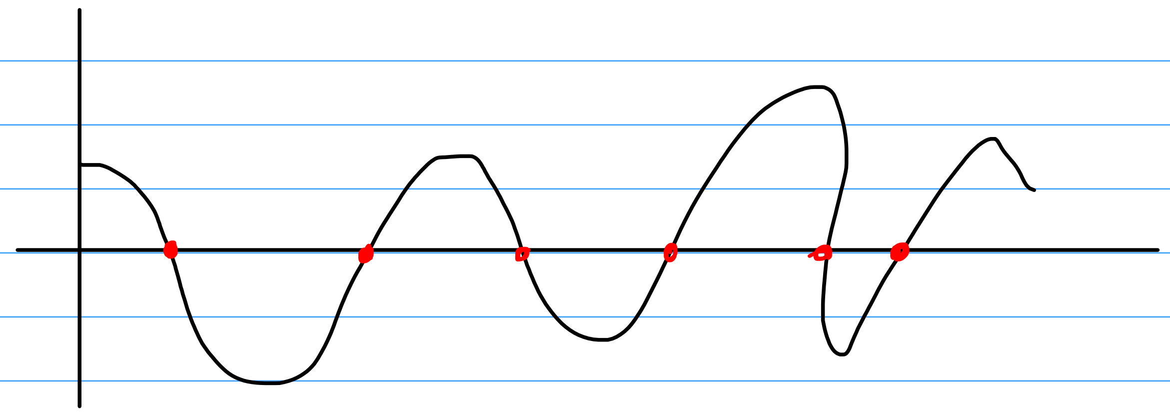
Define function:

$$\varphi(x) := f(x) - \underbrace{L_{\mathcal{T}} f(x)}_{\in \mathcal{P}_n} - c\,\underbrace{\omega(x)}_{\in \mathcal{P}_{n+1}} \quad \in C^{n+1}(I)$$

How many zeros does φ have at least?

$$\left.\begin{array}{lll} \varphi(t_j) = 0 & (\text{I.C}) & j = 0, \ldots, n \to n+1 \text{ zeros} \\ \varphi(t) = 0 & (\text{by def.}) & \end{array}\right\}$$

$\Rightarrow n+2$ <u>distinct</u> zeros



φ' has at least $n+1$ distinct zeros [mean value thm]

φ'' n distinct zeros

$\vdots$

$\varphi^{(n+1)}(x)$ at least 1 zero name it τ_t

$$\varphi^{(n+1)}(x) = f^{(n+1)}(x) - 0 - c\,(n+1)!$$

and

$$\varphi^{(n+1)}(\tau_t) = 0 = f^{(n+1)}(\tau_t) - c\,(n+1)!$$

$$c = \frac{f^{(n+1)}(\tau_t)}{(n+1)!} \qquad \Box$$

Use this pointwise estimate to get a global estimate:

$$\text{Thm. 6.1.44} \Rightarrow \boxed{\|f - \mathsf{L}_{\mathcal{T}} f\|_{L^\infty(I)} \le \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \max_{t\in I} |(t-t_0)\cdot\cdots\cdot(t-t_n)|} \,. \tag{6.1.50}$$

Example of $\sin(t)$ on $[0,\pi]$:

$$\|f^{(n)}\|_\infty \le 1$$

$$\Rightarrow \|f - L_T f\|_{L^\infty(I)} \le \frac{1}{(n+1)!} \underbrace{\max_{t\in[0,\pi]} \left|t\cdot\left(t-\frac{\pi}{n}\right)\left(t-\frac{2\pi}{n}\right)\cdots(t-\pi)\right|}_{\text{extremal at } \approx \frac{\pi}{2n}}$$

equidistant nodes & Lagrange interp.

$$\le \frac{1}{(n+1)!} \left|\frac{\pi}{2n}\left(\frac{\pi}{2n} - \frac{\pi}{n}\right)\cdots\left(\frac{\pi}{2n} - \pi\right)\right|$$

$$\le \frac{1}{(n+1)!}\left(\frac{\pi}{n}\right)^{n+1} \underbrace{\left|\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)\right|}_{\le n!}$$

$$\leq \frac{1}{n+1} \left(\frac{\pi}{n}\right)^{n+1}$$

faster than exponential

Example: $f(t) = \frac{1}{1+t^2}$ on $I = [-5,5]$

$$\|f^{(n)}\|_{L^\infty([-5,5])} \sim 2^n n! \quad \text{for } n \to \infty$$

Estimate (6.1.50) no longer guarantees convergence (i.e. blow-up possible)

[RHS of (6.1.50) roughly:

$$\sim 2^{n+1} (n+1)! \frac{1}{(n+1)!} n! \left(\frac{5}{n}\right)^{n+1} 2^n$$

$$= n! \left(\frac{20}{n}\right)^n \cdot \frac{10}{n}$$

By Stirling this grows exponentially:

$$\underbrace{n^{n+1/2} \sqrt{2\pi}\, e^{-n} \left(\frac{20}{n}\right)^n \frac{10}{n}}_{= \left(\frac{20}{e}\right)^n \sqrt{2\pi} \frac{10}{n^{1/2}}} \leq n! \left(\frac{20}{n}\right)^n \cdot \frac{10}{n}$$

$\left(\frac{20}{e}\right)^n$ exp. growth!

Note: there is also an L^2-estimate for $f \in C^{n+1}(I)$ and $\mathcal{T} := \{t_0, \dots, t_n\} \in I$

$$\|f - L_{\mathcal{T}} f\|_{L^2(I)} \leq \frac{2^{(n-1)/4} |I|^{n+1}}{\sqrt{n! (n+1)!}} \|f^{(n+1)}\|_{L^2(I)} .$$

6.1.3 Chebychev interpolation

Recall: RHS of (6.1.50) depended on

$$\|f^{(n+1)}\|_{L^\infty(I)} \quad \text{and}$$

we can't control

$$\max_{t \in I} |\omega(t)|$$

here we have some choice

New task: Given n and I

find nodes $t_0, \dots, t_n$ s.t.

$\boxed{\|\omega\|_{L^\infty(I)} \text{ minimal!}}$

Recall: $\omega(t) = \prod_{j=0}^{n} (t - t_j) \in \mathcal{P}_{n+1}$

leading coeff. is 1

Equivalent problem:

$\boxed{\text{Find } q \in \mathcal{P}_{n+1} \text{ with leading coeff. } 1 \text{ s.t. } \|q\|_{L^\infty(I)} \text{ minimal}}$

This implies that q has $n+1$ zeros in I:

Take $I = [-1, 1]$ and $q(t) := \prod_{j=0}^{n} (t - t_j)$

with $t_0 < -1$,

Then, define

$$p(t) := (t+1) \cdot (t - t_1) \dots (t - t_n)$$

$\Rightarrow \quad |p(t)| < |q(t)| \quad \forall\, t \in I = [-1, 1]$

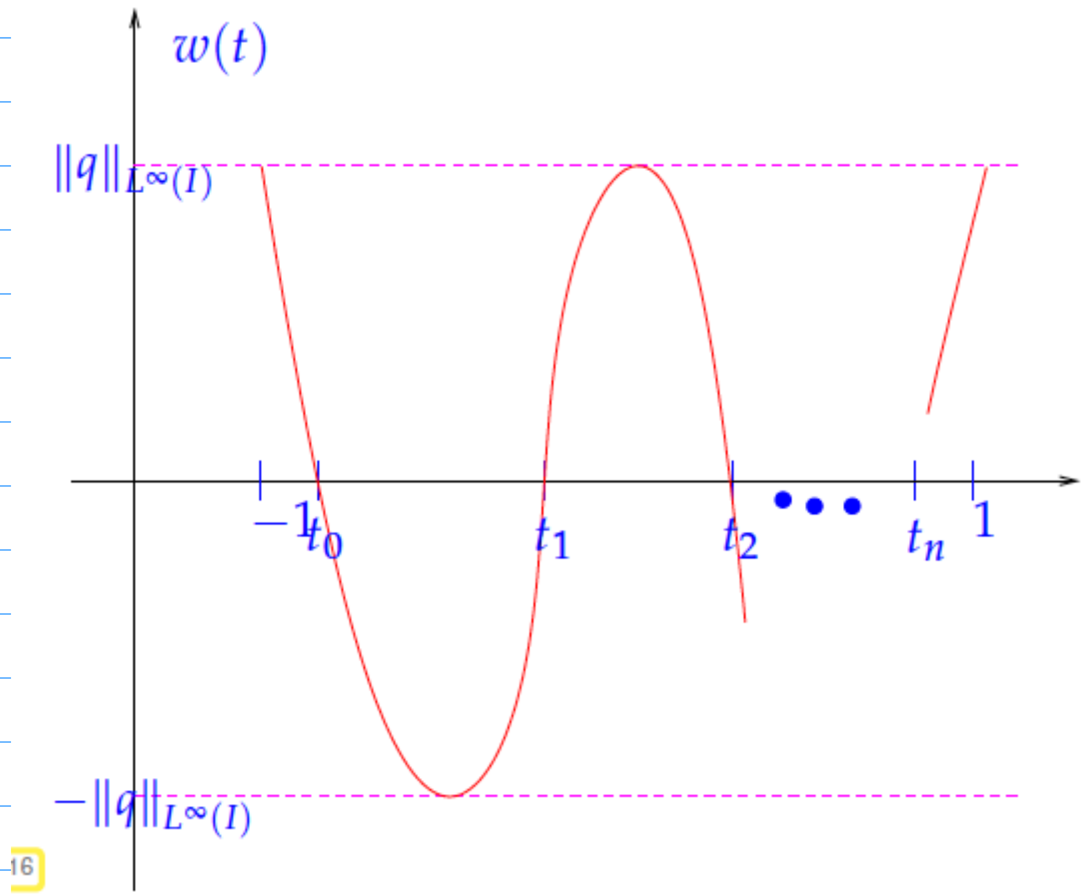
$\Rightarrow \quad \|p\|_{L^\infty(I)} < \|q\|_{L^\infty(I)}$ $\lightning$ to q having min norm

Recipe once q is found:

Take nodes $t_0, \dots, t_n$ to be the zeros of q.

As we will see: For $q \in \mathcal{P}_{n+1}$ with $\|q\|_{L^\infty(I)}$ min.:

$\|q\|_{L^\infty}$ will be attained at edges of the boundary & local extrema of q (in abs. value):



Note: Goal here is an a-priori choice of nodes without using information on f!
A-posteriori methods also exist.

Can we find the minimizing q?

$\rightsquigarrow$ Chebychev polynomials

Definition 6.1.76. Chebychev polynomials $\to$ [?, Ch. 32]

The n^{th} Chebychev polynomial is $T_n(t) := \cos(n \arccos t)$, $\quad -1 \le t \le 1, n \in \mathbb{N}$.

$|T_n(t)| \le 1 \qquad \forall t \in [-1,1]$

Are there actual polynomials?

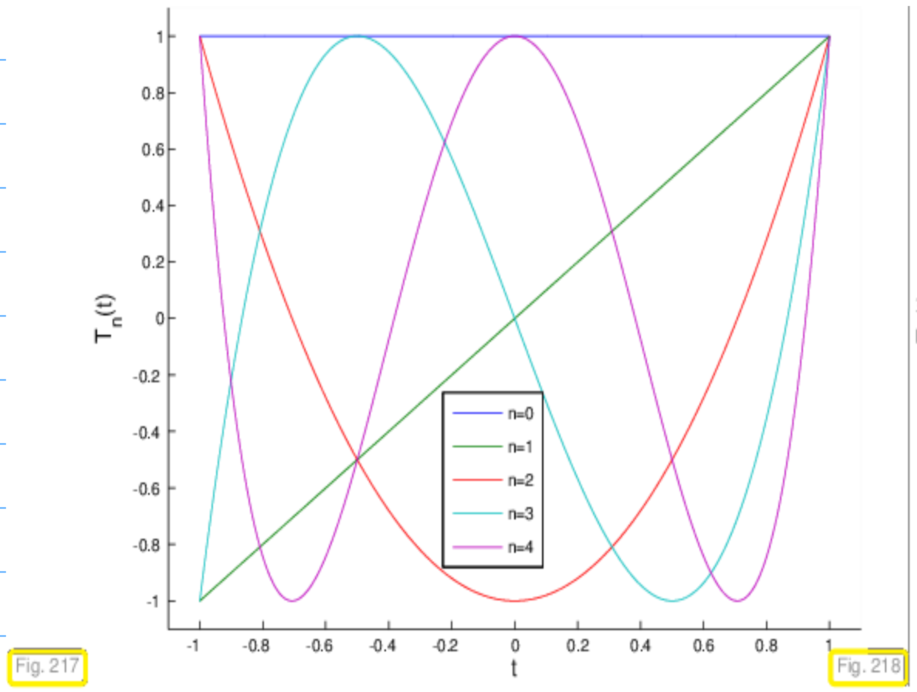
Theorem 6.1.77. 3-term recursion for Chebychev polynomials $\to$ [?, (32.2)]

The function T_n defined in Def. 6.1.76 satisfy the 3-term recursion

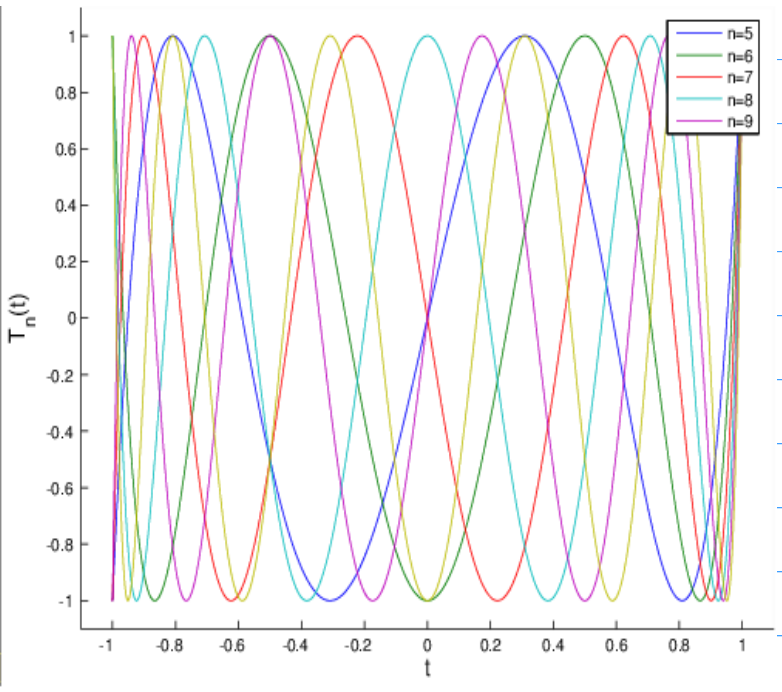
$$T_{n+1}(t) = \underline{2t\,T_n(t) - T_{n-1}(t)} \;, \quad T_0 \equiv 1 \,, \quad T_1(t) = t \,, \quad n \in \mathbb{N} \,. \tag{6.1.78}$$

If $T_{n-1} \in \mathcal{P}_{n-1}$, $T_n \in \mathcal{P}_n \Rightarrow T_{n+1} \in \mathcal{P}_{n+1}$

$T_0 \in \mathcal{P}_0$, $T_1 \in \mathcal{P}_1 \Rightarrow$ by induction: $T_n \in \mathcal{P}_n \quad \forall n \in \mathbb{N}$.



Chebychev polynomials $T_0, \ldots, T_4$



Chebychev polynomials $T_5, \ldots, T_9$

$$T_2(t) = 2t^2 - 1$$

$$T_3(t) = 4t^3 - 2t - t$$

From the recursion: leading coefficient of T_n: 2^{n-1}

Zeros of $T_n(t)$:

$$n \arccos(t_j) = \frac{2j+1}{2}\pi \qquad j = 0, \ldots, n$$

$$\arccos(t_j) = \frac{2j+1}{2n}\pi$$

$$t_j = \cos\left(\frac{2j+1}{2n}\pi\right)$$ Chebychev nodes

Is this choice optimal?

Theorem 6.1.82. Minimax property of the Chebychev polynomials [?, Section 7.1.4.], [?, Thm. 32.2]

The polynomials T_n from Def. 6.1.76 minimize the supremum norm in the following sense:

$$\|T_n\|_{L^\infty([-1,1])} = \inf\{\|p\|_{L^\infty([-1,1])} : p \in \mathcal{P}_n, p(t) = 2^{n-1}t^n + \cdots\}, \quad \forall n \in \mathbb{N}.$$

Proof:

1. Local extrema of T_n?

$$|\cos x| = 1 \iff x = j\pi \text{ for some } j \in \mathbb{Z}$$

$$|T_n(\tilde{t}_j)| = 1 \iff n \arccos(\tilde{t}_j) = j\pi$$

$$\tilde{t}_j = \cos\left(\frac{j\pi}{n}\right) \qquad j = 0, \ldots, n$$

$\Rightarrow$ $n+1$ local extrema of T_n in $I = [-1,1]$.

Suppose $\exists\, q \in \mathbb{P}_n$ with leading coeff. 2^{n-1} s.t.

$$\|q\|_{L^\infty(I)} < \|T_n\|_{L^\infty(I)} = 1$$

$\Rightarrow$ • If $\tilde{t}_j$ local minimum of T_n: $T_n(\tilde{t}_j) = -1$

$$\Rightarrow (T_n - q)(\tilde{t}_j) < 0$$

• If $\tilde{t}_j$ local maximum of T_n: $T_n(\tilde{t}_j) = 1$

$$\Rightarrow (T_n - q)(\tilde{t}_j) > 0$$

# of local extrema $n+1$:

$T_n - q$ changes sign at least n times on $[-1,1]$

$T_n - q$ has at least n zeros

T_n and q have the same leading coefficient

$\Rightarrow T_n - q \in \mathbb{P}_{n-1}$

$T_n - q \equiv 0$ $\lightning$ to $\|q\|_{L^\infty(I)} < \|T_n\|_{L^\infty(I)}$ $\square$

This result implies: nodal polynomial

$$\omega(t) = \prod_{j=0}^{n} (t - t_j)$$

with $\|\omega\|_{L^\infty(I)}$ minimal is found by

$$\boxed{\omega(t) := 2^{-n}\, T_{n+1}(t)}$$

and t_j are zeros of T_{n+1} (Chebychev nodes)

$$t_j = \cos\left(\frac{2j+1}{2n}\pi\right) \qquad j = 0, \dots, n$$

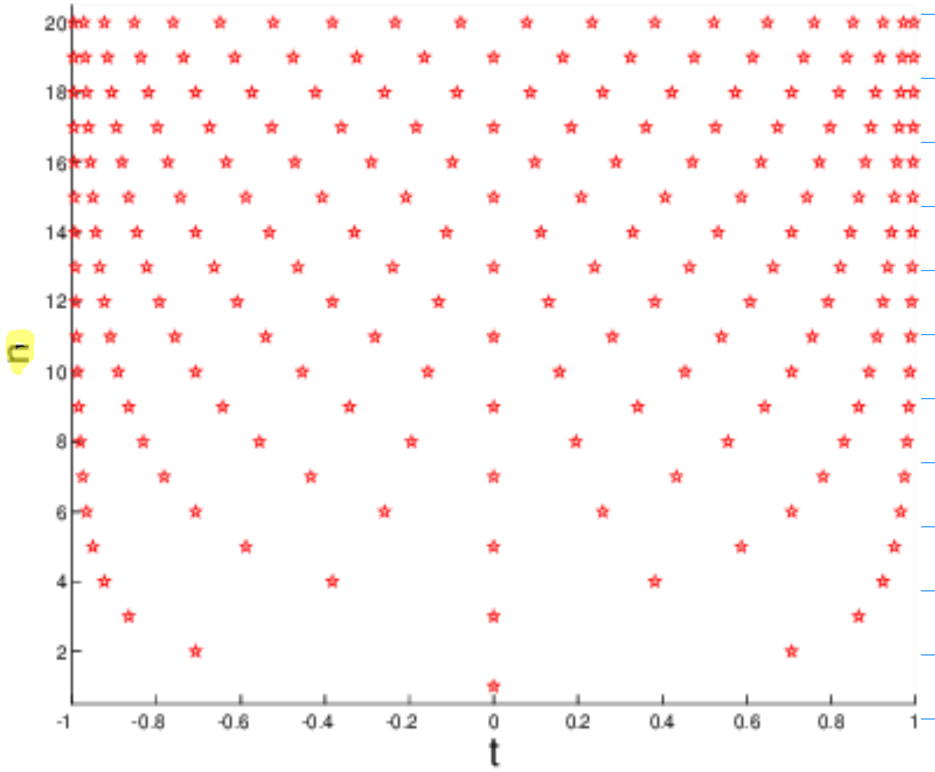


Fig. 219

Distribution of Chebyshev nodes: more dense at boundaries as n increases

$I = [-1, 1]$

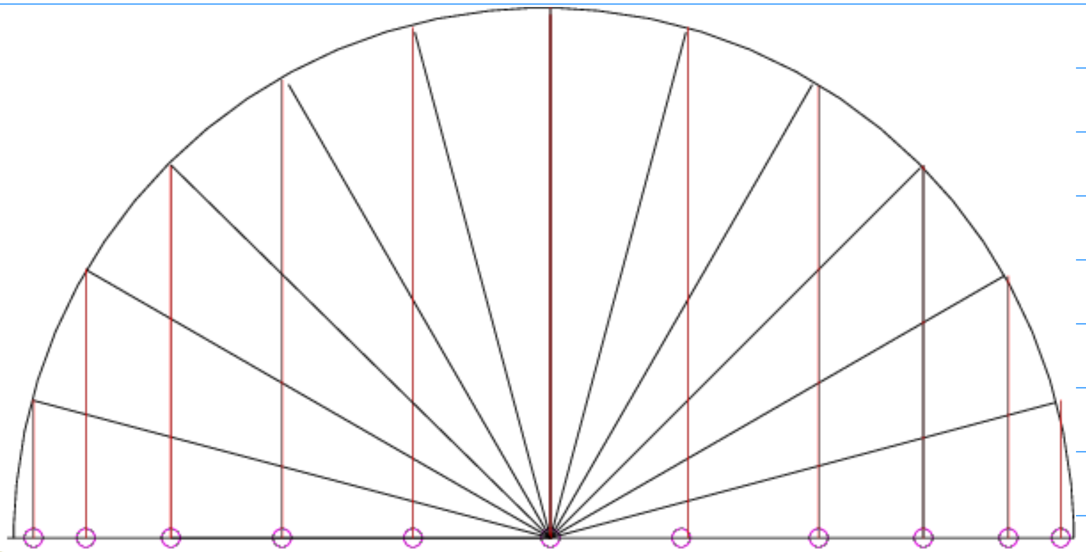


Fig. 220

equidistant on circle
→ more dense at edges

With choice $\omega(t) = 2^{-n} T_{n+1}(t)$

what can we say about interpolation error?

Now $\|\omega\|_{L^\infty([-1,1])} = 2^{-n}$

$f \in C^{n+1}([-1,1])$

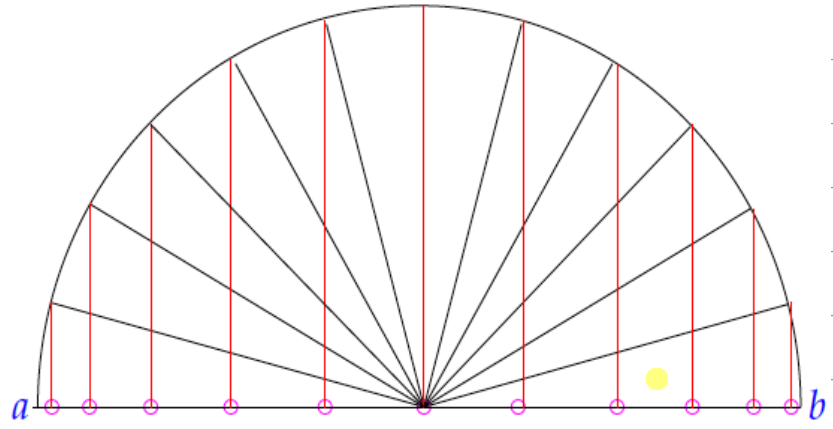
$$\|f - L_T f\|_{L^\infty([-1,1])} \leq \boxed{\frac{2^{-n}}{(n+1)!}} \|f^{(n+1)}\|_{L^\infty([-1,1])}$$

On arbitrary intervals $[a,b]$:

transform nodes with affine linear transformation:

$$[-1, 1] \to [a,b]$$

$$\hat{t} \mapsto a + \frac{b-a}{2}(\hat{t} + 1) \qquad \hat{t} \in [-1,1]$$



The Chebychev nodes in the interval $I = [a,b]$ are

$$t_k := a + \tfrac{1}{2}(b-a)\left(\cos\left(\frac{2k+1}{2(n+1)}\pi\right)+1\right), \qquad (6.1.87)$$

$$k = 0,\dots,n.$$

Interpolation error estimate for general $I=[a,b]$:

$$\|f - \mathsf{I}_{\mathcal{T}}(f)\|_{L^\infty(I)} = \|\overset{\text{pullback}}{\widehat{f}} - \mathsf{I}_{\widehat{\mathcal{T}}}(\widehat{f})\|_{L^\infty([-1,1])} \le \frac{2^{-n}}{(n+1)!}\left\|\frac{d^{n+1}\widehat{f}}{d\widehat{t}^{n+1}}\right\|_{L^\infty([-1,1])}$$
$$\le \frac{2^{-2n-1}}{(n+1)!}|I|^{n+1}\left\|f^{(n+1)}\right\|_{L^\infty(I)}. \qquad (6.1.88)$$

much better estimate than for equidistant nodes

But: still possible divergence if $\|f^{(n)}\|_{L^\infty(I)}$ grows too fast.

Example: $\frac{1}{1+t^2}$ $\quad t \in [-5,5]$

$$\|f^{(n+1)}\|_{L^\infty([-5,5])} \sim 2^{n+1}(n+1)!$$

Still RHS blows up: $\frac{2^{-2n-1}}{\cancel{(n+1)!}}\, 10^{n+1}\, 2^{n+1}\, \cancel{(n+1)!}$

$= 5^n \cdot 10$

exp. growth

BUT: not necessarily divergence

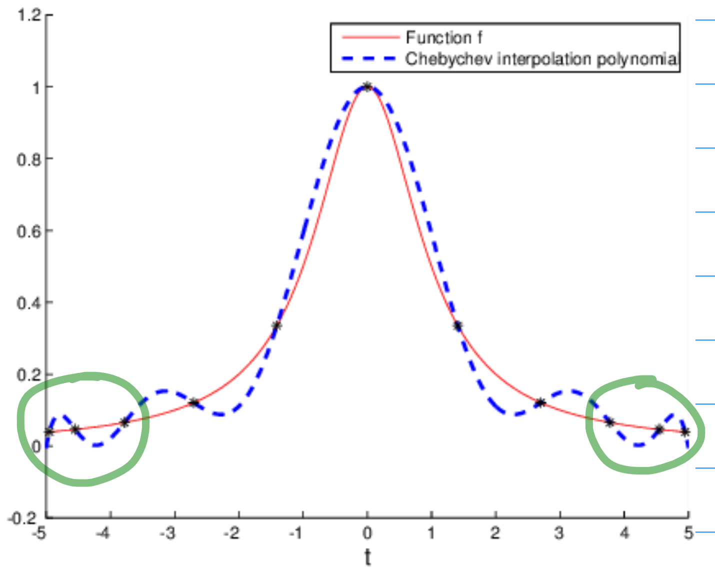


Fig. 222

Chebychev nodes $\quad n=10$

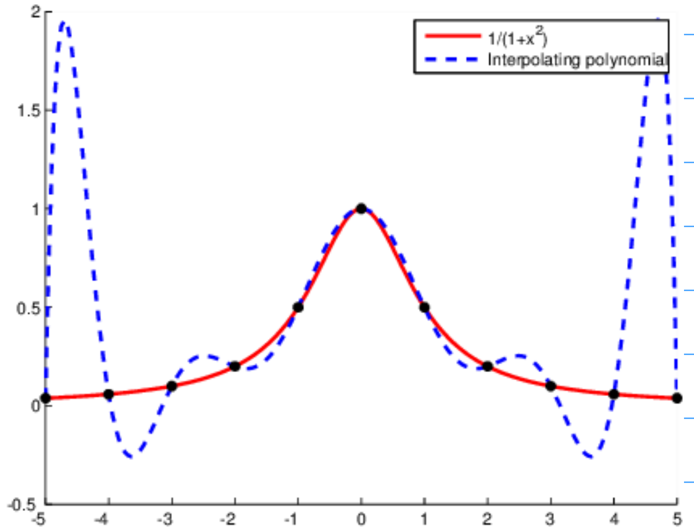


Fig. 221

Equidistant nodes

Estimates independent of $\|f^{(n)}\|_\infty$

Concept: Lebesgue constant [quality measure for polyn. interp. scheme]

Given $\mathcal{T} = \{t_0, \dots, t_n\}$

$$I_{\mathcal{T}}: \mathbb{R}^{n+1} \to \mathcal{P}_n$$

$$I_{\mathcal{T}}((f(t_0), \dots, f(t_n)) = L_{\mathcal{T}} f$$

Lebesgue constant: $\lambda_{\mathcal{T}} := \sup_{y \in \mathbb{R}^{n+1}} \frac{\|I_{\mathcal{T}} y\|_\infty}{\|y\|_\infty}$

$$L_{\mathcal{T}}: C^0(I) \to \mathcal{P}_n$$

$$f \mapsto L_{\mathcal{T}} f = I_{\mathcal{T}}(\overbrace{(f(t_0), \dots, f(t_n))}^{=z})$$

$$\|L_{\mathcal{T}} f\|_{L^\infty(I)} = \|I_{\mathcal{T}}(z)\|_{L^\infty(I)}$$

$$= \frac{\|I_{\mathcal{T}}(z)\|_\infty}{\|z\|_\infty} \cdot \|z\|_\infty \leq \lambda_{\mathcal{T}} \|z\|_\infty$$

$$\leq \lambda_{\mathcal{T}} \|f\|_{L^\infty(I)}$$

$$\Rightarrow \|L_{\mathcal{T}} f\|_{L^\infty(I)} \leq \lambda_{\mathcal{T}} \|f\|_{L^\infty(I)}$$

[$\lambda_{\mathcal{T}}$: norm of the operator $I_{\mathcal{T}}$ in L^∞-sense]

More explicitly: $\lambda_{\mathcal{T}} = \max_{t \in I} \sum_{j=0}^{n} |L_j(t)|$

Lagrange polynomials

Implication for interpolation error:

$$\|f - L_{\mathcal{T}} f\|_{L^\infty(I)} = \|f - p - (L_{\mathcal{T}} f - p)\|_{L^\infty(I)} \quad \text{for any } p \in \mathcal{P}_n$$

$$= \|f - p - L_{\mathcal{T}}(f-p)\|_{L^\infty(I)}$$

$L_{\mathcal{T}} p = p$

linear

$$\underset{\substack{\uparrow \\ \text{triangle} \\ \text{ineq.}}}{\leq} \|f-p\|_\infty + \underbrace{\|L_T(f-p)\|_\infty}$$

$$\leq \|f-p\|_\infty + \lambda_T \|f-p\|_\infty$$

$$= \boxed{(1+\lambda_T)\|f-p\|_\infty}$$

$$\|f - \mathsf{L}_\mathcal{T} f\|_{L^\infty(I)} \leq (1+\lambda_\mathcal{T}) \underbrace{\inf_{p \in \mathcal{P}_n} \|f-p\|_{L^\infty(I)}}_{\text{best approximation error}} \quad \forall f \in C^0(I)\,, \tag{6.1.61}$$

interpolation error is <u>at most</u> $1+\lambda_T$ worse than the best approximation error!

(6.1.61) is a special case of Lebesgue's lemma in approximation theory:

Given $(X, \|\cdot\|)$ normed vector space, U subspace of X and P is a linear projection on U,

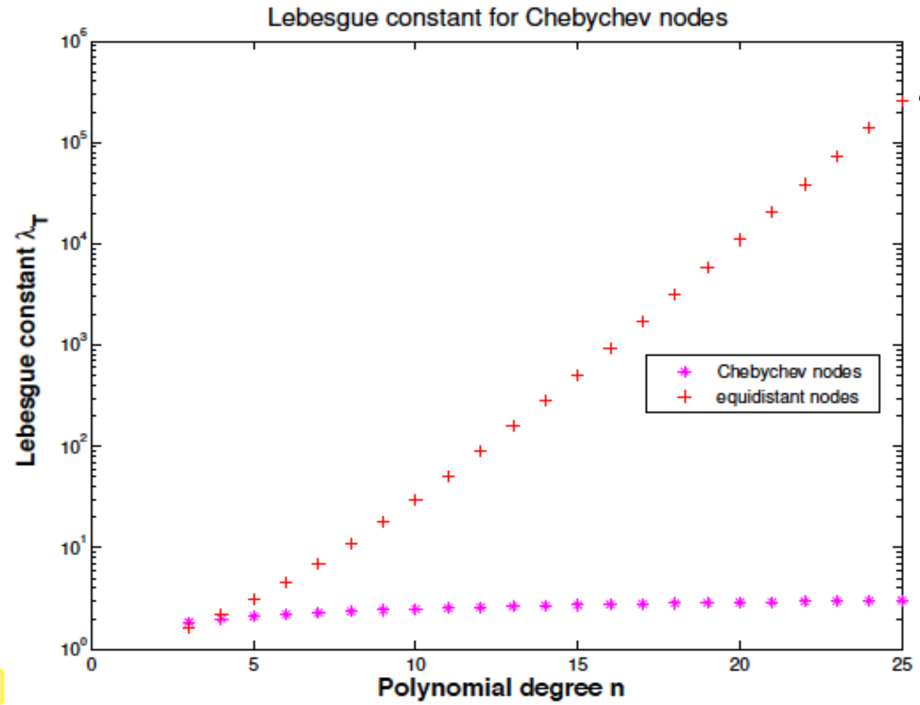
$P: X \to U$

Then $\|x - Px\| \leq (1 + \|P\|) \inf_{u \in X} \|x-u\| \quad \forall x \in X$

Quantify λ_T ?

Equidistant nodes: $\lambda_T \geq Ce^{n/2}$ (at least exp. growth)

Chebychev nodes: $\lambda_T \leq \frac{2}{\pi} \log(n+1) + 1$ (at most log growth)



← equidistant nodes

lin-log-plot

← Chebychev nodes

Combine general estimate (6.1.61) with Jackson's thm:

$$\|f - L_T f\|_{L^\infty([-1,1])} \le (1+\lambda_T) \inf_{p\in\mathcal{P}_n} \|f-p\|_{L^\infty([-1,1])}$$

$$\le (1+\lambda_T)\left(1+\frac{\pi^2}{2}\right)^r \frac{(n-r)!}{n!} \|f^{(r)}\|_{L^\infty([-1,1])}$$

for $f\in C^r([-1,1])$

For Chebychev interpolation:

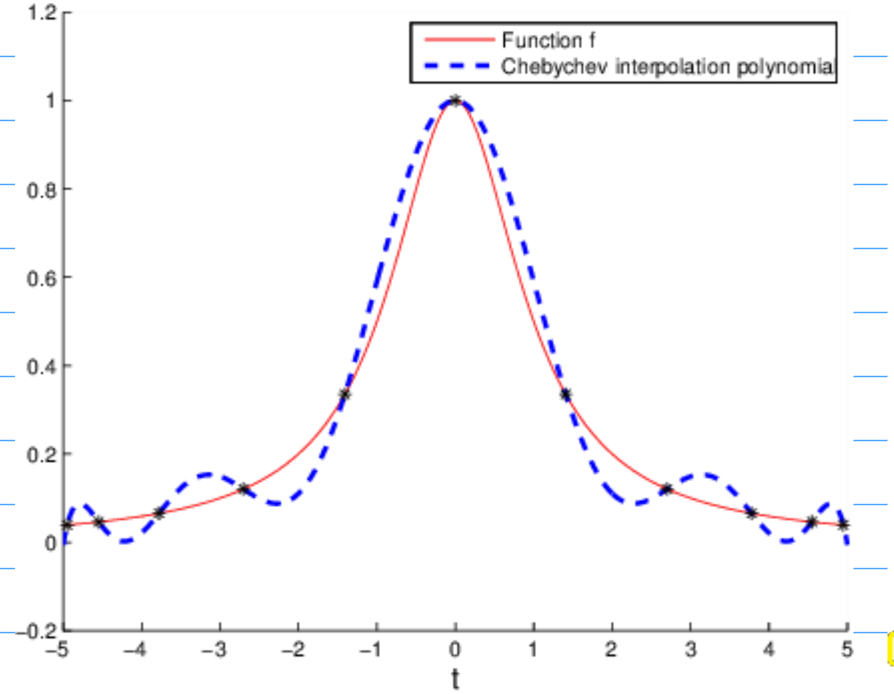
$$\|f - \mathsf{L}_\mathcal{T} f\|_{L^\infty([-1,1])} \le \underbrace{(2/\pi \log(1+n)+2)}_{1+\lambda_T}\underbrace{(1+\pi^2/2)^r \frac{(n-r)!}{n!}}_{\le C(r)\, n^{-r}} \left\|f^{(r)}\right\|_{L^\infty([-1,1])}$$

alg. convergence

for fixed r and $n\to\infty$.

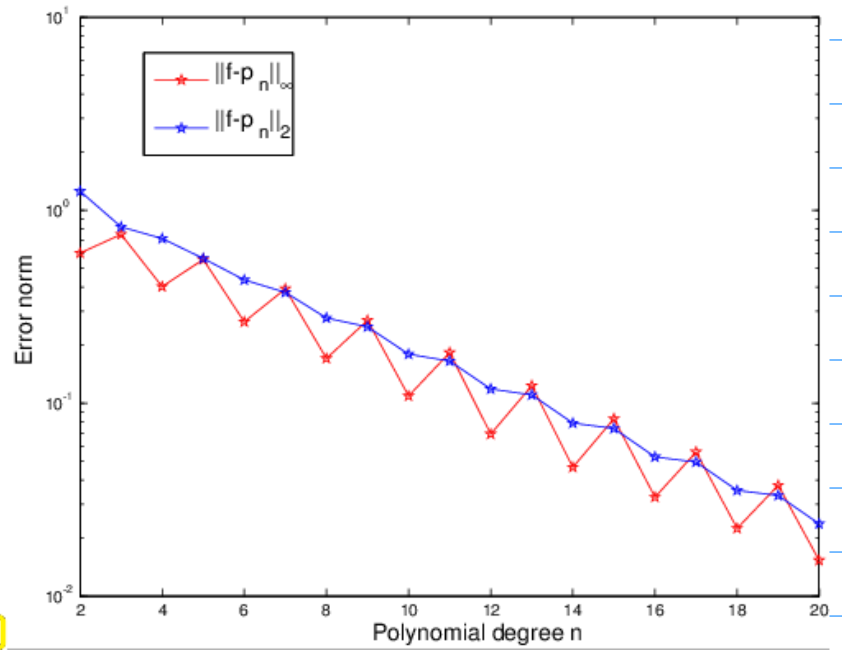
Examples with Chebychev interpolation:

① $f(t) = \dfrac{1}{1+t^2}$ $\qquad t\in[-5,5]$



$n=10$

Fig. 223



roughly: exp. convergence

mostly true when $f\in C^\infty$

(2) Hat function only C^0

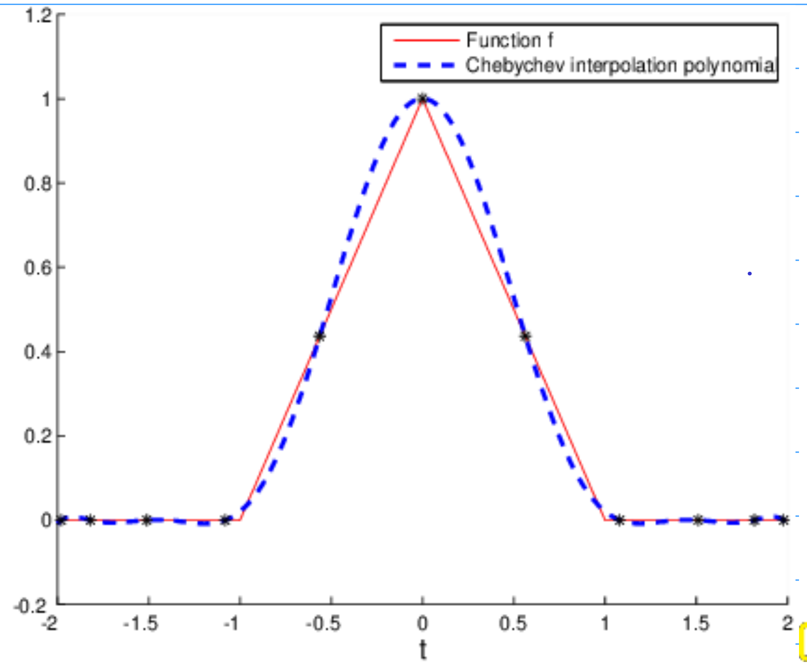


Fig. 224

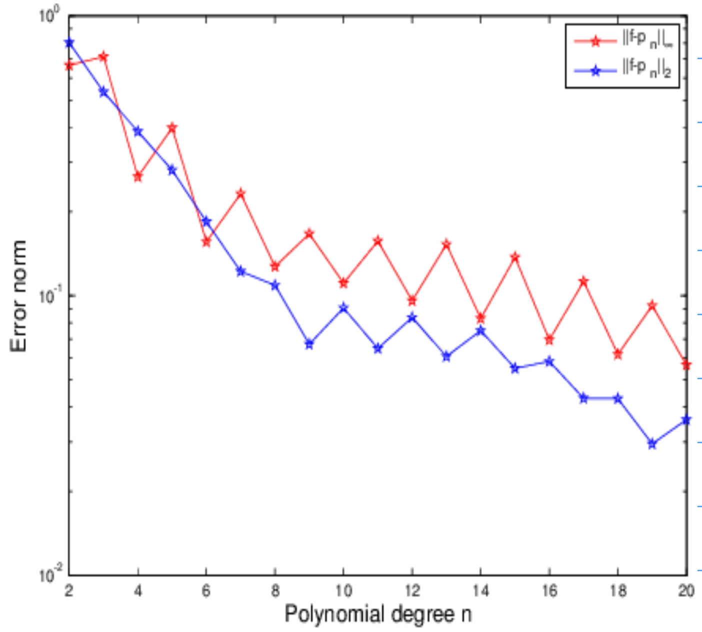


Fig. 225

$n=10$

still: convergence (not exp.)

(3) $f(t) = \begin{cases} \frac{1}{2}(1+\cos(\pi t)) & |t|<1 \\ 0 & |t|\in[1,2] \end{cases}$ $\quad I=[-2,2]$

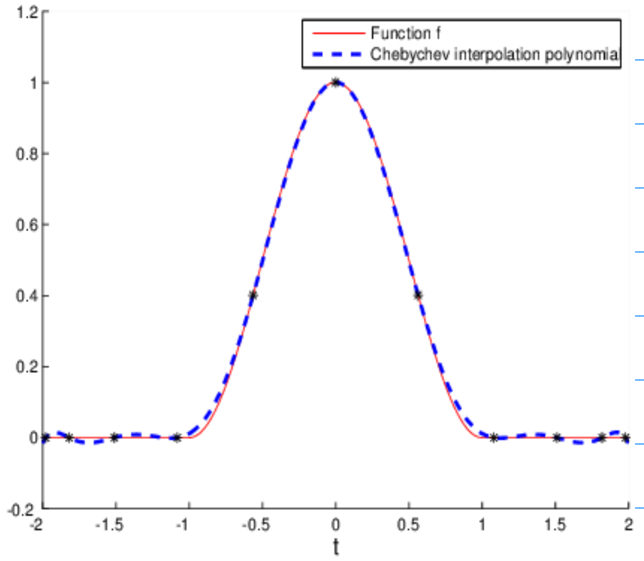


Fig. 227

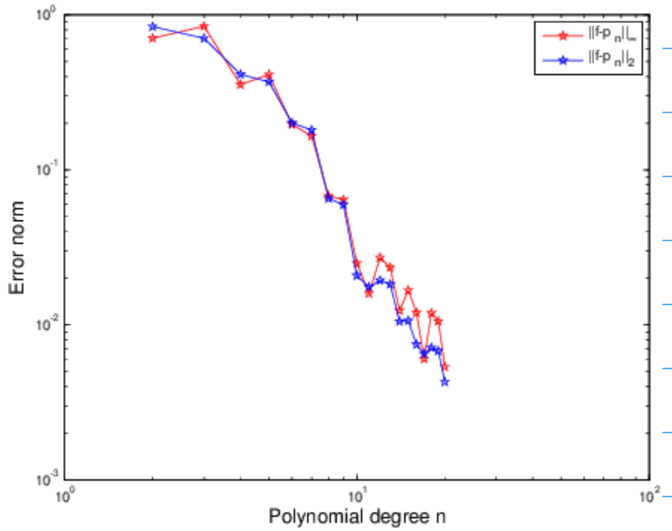


Fig. 228

~ alg. convergence

6.1.3.3 Implementation of Chebychev interpolation

Idea: interpolant $p \in \mathcal{P}_n$ as

$$p(t) = \sum_{j=0}^{n} \alpha_j \underset{\substack{\uparrow \\ \text{Chebychev polynomial} \\ \text{basis of } \mathcal{P}_n}}{T_j(t)}$$

$\deg T_j = j \Rightarrow \{T_0, \dots, T_n\}$ basis of $\mathcal{P}_n$

① Efficient evaluation of p given coefficients α_j

Idea: Use the 3-term recursion of Chebychev polynomials:

$$T_j(t) = 2t\, T_{j-1}(t) - T_{j-2}(t) \qquad j = 2, \dots, n$$

$$p(x) = \sum_{j=0}^{n-1} \alpha_j T_j(x) + \alpha_n T_n(x)$$

$$\overset{(6.1.78)}{=} \underbrace{\sum_{j=0}^{n-1} \alpha_j T_j(x)}_{*} + \underbrace{\alpha_n (2x T_{n-1}(x) - T_{n-2}(x))}$$

$$= \underbrace{\sum_{j=0}^{n-3} \alpha_j T_j(x) + (\underline{\alpha_{n-2} - \alpha_n}) T_{n-2}(x) + (\underline{\alpha_{n-1} + 2x\alpha_n}) T_{n-1}(x)}_{\text{another Chebychev expansion}} .$$

$(*)$: $\sum_{j=0}^{n-3} \alpha_j T_j(x) + \underbrace{\alpha_{n-2} T_{n-2}(x) + \alpha_{n-1} T_{n-1}(x)}$

$$p(x) = \sum_{j=0}^{n-1} \tilde{\alpha}_j T_j(x) \quad \text{with} \quad \tilde{\alpha}_j = \begin{cases} \alpha_j + 2x\alpha_{j+1} & \text{, if } j = n-1 , \\ \alpha_j - \alpha_{j+2} & \text{, if } j = n-2 , \\ \alpha_j & \text{else.} \end{cases} \qquad (6.1.103)$$

C++11 code 6.1.104: Recursive evaluation of Chebychev expansion (6.1.101)

```
// Recursive evaluation of a polynomial p = \sum_{j=1}^{n+1} a_j T_{j-1} at point x
// based on (6.1.103)
// IN : Vector of coefficients a
// evaluation point x
// OUT: Value at point x
double recclenshaw(const VectorXd& a, const double x) {
  const VectorXd::Index n = a.size() - 1;
  if      (n == 0) return a(0);              // Constant polynomial
  else if (n == 1) return (x*a(1) + a(0)); // Value α1 * x + α0
  else {
    VectorXd new_a(n);
    new_a << a.head(n - 2), a(n - 2) - a(n), a(n - 1)+ 2*x*a(n);
    return recclenshaw(new_a,x); // recursion
  }
}
```

Complexity: $O(n)$.