SINGULAR VALUE DECOMPOSITION

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Abstract. We try to motivate the proof of the singular value decompostion for a linear map between two finite dimensional $\mathbb{R}$ vector spaces and argue that the proof follows with elementary trickery from the right formulation of the question.

We first motivate the statement. Assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Let $r = \operatorname{rank} T$, i.e. r is the dimension of the image of T. Let $(w_1, \ldots, w_r)$ be any basis of $\operatorname{im} T$. By definition there are $v_1, \ldots, v_r \in \mathbb{R}$ such that $Tv_i = w_i$ and thus $(v_1, \ldots, v_r)$ is a linearly independent set. One easily checks that $\ker T \cap \langle v_1, \ldots, v_r \rangle = \{0\}$, and hence we can find an extension $(v_1, \ldots, v_n)$ to a basis of $\mathbb{R}^n$ such that the matrix representing T with respect to this basis and any extension of the w_i to a basis of $\mathbb{R}^m$ is of the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r denotes the $r \times r$ identity matrix. If we normalize the v_i with respect to the Euclidean metric on $\mathbb{R}^n$, then we would have to replace the identity matrix in the top left corner by a diagonal matrix, whose diagonal entries are the original length of the vectors v_i.

The singular value decomposition is nothing but a refined version of this statement. Its proof is a bit more intricate, as we can certainly start with an orthonormal basis of $\operatorname{im} T$ and we can of course extend it to an orthonormal basis of $\mathbb{R}^m$, but there is no reason why the v_i should be orthogonal. However the singular value decomposition states that there exists an orthonormal basis of $\mathbb{R}^m$ such that the corresponding v_i can be chosen orthonormally.

There are two main ingredients:

- We want to somehow single out a designated basis of $\operatorname{im} T$ for which the v_i could be orthogonal, but ex ante it is not clear, where this would come from. Let T^* denote the adjoint to T. Then TT^* is self-adjoint non-negative definite and hence $\mathbb{R}^m$ has an orthonormal basis consisting of eigenvectors of TT^*. As $\operatorname{im} TT^* = \operatorname{im} T$, there is a very special basis of $\operatorname{im} T$, namely the eigenvectors of TT^* for positive eigenvalues.
- The map T descends to a bijection between $(\ker T)^\perp$ and $\operatorname{im} T$, hence given any $w \in \operatorname{im} T$, there is a unique $v \in (\ker T)^\perp$ such that $Tv = w$. Note that $T^*\mathbb{R}^m \subseteq (\ker T)^\perp$, and as the restriction of TT^* is diagonalizable, so is its inverse. Hence one expects that for the unique $v_i \in (\ker T)^\perp$ satisfying $Tv_i = w_i$ for some eigenvector $w_i \in \operatorname{im} T$ of TT^*, one also has that T^*w_i is a multiple of v_i. It will however turn out that in the proof it is much easier to define the v_i using T^* directly.

We now set out to give a rigorous proof of the following

Theorem 0.1 (Singular Value Decomposition)**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ linear and $r = \operatorname{rank} T \geq 1$. Then there exist $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and orthonormal bases $(v_i)_{i=1}^n$*

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and $(w_j)_{j=1}^m$ *of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively, so that* $Tv_i = \sigma_i w_i$ *whenever* $1 \leq i \leq r$ *and* $v_i \in \ker T$ *otherwise.*

Proof. As TT^* is non-negative definite self-adjoint, there exists an orthonormal basis $(w_j)_{j=1}^m$ of $\mathbb{R}^m$ consisting of eigenvectors of TT^*. For all $1 \leq j \leq m$ let $\lambda_j \in \mathbb{R}$ such that $TT^* w_j = \lambda_j w_j$. After permutation of the elements of the basis, we can assume without loss of generality that $\lambda_1 \geq \cdots \geq \lambda_\rho > 0 = \lambda_{\rho+1} = \cdots = \lambda_m$, where $\rho = \operatorname{rank} TT^*$. In particular, $(w_j)_{j=1}^\rho$ is an orthonormal basis of $\operatorname{im} TT^*$.

We show now that $\operatorname{im} TT^* = \operatorname{im} T$. The inclusion $\operatorname{im} TT^* \subseteq \operatorname{im} T$ is immediate, and using the assumption of finite dimensionality, ist suffices to show that $\operatorname{rank} TT^* = \operatorname{rank} T$. First we show that $\ker TT^* = \ker T^*$. Again, one inclusion is clear, as $T^* w = 0 \implies TT^* w = 0$. For the opposite inclusion, assume that $w \in \ker TT^*$. Then

$$0 = \langle TT^* w, w \rangle = \langle T^* w, T^* w \rangle \implies T^* w = 0$$

and hence follows $\ker TT^* \subseteq \ker T^*$. Hence the dimension formula implies that

$$\begin{aligned} \operatorname{rank} TT^* &= m - \dim(\ker TT^*) = m - \dim(\ker T^*) \\ &= \operatorname{rank} T^* = \operatorname{rank} T \end{aligned}$$

as desired. It follows, that $\rho = r$ and that $(w_j)_{j=1}^r$ is an orthonormal basis of $\operatorname{im} T$.

For $1 \leq i \leq r$ set $\tilde{v}_i = T^* w_i$, then for all $1 \leq i, j \leq r$ holds

$$\lambda_i \delta_{ij} = \langle TT^* w_i, w_j \rangle = \langle \tilde{v}_i, \tilde{v}_j \rangle$$

and thus $(\tilde{v}_j)_{j=1}^r$ form an orthogonal family of non-zero vectors. In particular, they are linearly independent. Let $v \in \ker T$, then we get for $1 \leq i \leq r$

$$0 = \langle Tv, w_i \rangle = \langle v, \tilde{v}_i \rangle$$

and thus $(\ker T) \perp \langle \tilde{v}_1, \ldots, \tilde{v}_r \rangle$, so that using $n = \dim(\ker T) + r$ there exists an extension $(\tilde{v}_i)_{i=1}^n$ to an orthogonal basis of $\mathbb{R}^n$. Let $v_i = \frac{1}{\|\tilde{v}_i\|} \tilde{v}_i$ for all $1 \leq i \leq n$, then it follows that

$$T\Big(\sum_{i=1}^n \alpha_i v_i\Big) = \sum_{i=1}^n \alpha_i T v_i = \sum_{i=1}^r \alpha_i \sqrt{\lambda_i} w_i$$

and setting $\sigma_i = \sqrt{\lambda_i}$ the desired statement follows. □

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