D-MATH | Algebra I | HS17

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Assignment 2

Category theory, first definitions on Rings

1. Prove that a morphism in the category of sets is an isomorphism if and only if it is a bijective map.

2. Let $\mathcal{C}$ be a category and A an object of $\mathcal{C}$. Define F_A from $\mathcal{C}$ to sets by

$$\forall B \text{ object of } \mathcal{C},\ \ F_A(B) := \mathrm{Hom}_{\mathcal{C}}(A,B)$$
$$\forall f \in \mathrm{Hom}_{\mathcal{C}}(B,C),\ \ F_A(f) := \left(\begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C) \\ g \mapsto f \circ g \end{array} \right).$$

   Prove that F_A is a functor (it is called the *functor represented by* A).

3. We want to define a category $\mathcal{C}$ as follows:

   - An object (X, Y, f) of $\mathcal{C}$ is given by two sets X and Y and a map $f : X \longrightarrow Y$.
   - A morphism $(u, v) \in \mathrm{Hom}_{\mathcal{C}}((X, Y, f), (X', Y', f'))$ is given by maps $u : X \longrightarrow X'$ and $v : Y \longrightarrow Y'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

   (a) Define composition of morphisms so that $\mathcal{C}$ is indeed a category.

   (b) Prove that F from $\mathcal{C}$ to sets defined by $F((X, Y, f)) = X$ and $F((u, v)) = u$ is a functor.

4. Let R and S be two rings and $f : R \longrightarrow S$ a map between them. Prove that f is a ring isomorphism if and only if it is ring homomorphism and it is bijective.

5. (a) Compute the units of $\mathbb{Z}[i]$.

   (b) (*Euclidean division in* $\mathbb{Z}[i]$) Let $z, w \in \mathbb{Z}[i] \smallsetminus \{0\}$. Prove that there exist $q, r \in \mathbb{Z}[i]$ such that $z = q \cdot w + r$ and $|r| < |w|$. [*Hint:* Define $q \in \mathbb{Z}[i]$ such that it is a good approximation of $\frac{z}{w} \in \mathbb{C}$.]

6. Let $F(\mathbb{R}, \mathbb{C})$ the set of functions $\mathbb{R} \longrightarrow \mathbb{C}$. Denote by $C(\mathbb{R}, \mathbb{C})$ the subset of continuous functions and by $C_0(\mathbb{R}, \mathbb{C})$ the subset of continuous bounded functions.

   (a) Check that $F(\mathbb{R}, \mathbb{C})$, endowed with pointwise sum and multiplication, is a commutative ring. Find $F(\mathbb{R}, \mathbb{C})^\times$.

   (b) Prove that $C_0(\mathbb{R}, \mathbb{C})$ and $C(\mathbb{R}, \mathbb{C})$ are subrings of $F(\mathbb{R}, \mathbb{C})$.

   (c) Determine $C(\mathbb{R}, \mathbb{C})^\times$ and $C_0(\mathbb{R}, \mathbb{C})^\times$.

   (d) Is $C_0(\mathbb{R}, \mathbb{C})$ an integral domain?

   (e) Which of the following maps are ring homomorphisms?

      i. $\varphi : C_0(\mathbb{R}, \mathbb{C}) \longrightarrow \mathbb{C}$, sending $f \mapsto f(1)$;

      ii. $\psi : C_0(\mathbb{R}, \mathbb{C}) \longrightarrow \mathbb{R}$, sending $f \mapsto \sup_{x \in \mathbb{R}} |f(x)|$;

      iii. $\eta : C(\mathbb{R}, \mathbb{C}) \longrightarrow \mathbb{R}$, sending $f \mapsto \mathrm{Re}(f(0))$;

      iv. $\theta : \mathbb{Z} \longrightarrow F(\mathbb{R}, \mathbb{C})$ sending $n \in \mathbb{Z}$ to the constant function with value n.

7. Let $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ be the field with two elements $0, 1$. Define

$$R := \left\{ \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} : a, b \in \mathbb{F}_2 \right\}.$$

   (a) Prove that R is a commutative ring under the usual matrix sum and multiplication.

   (b) Prove that R is a field with exactly four elements.

8. Let R be a finite integral domain. Prove that R is a field. [*Hint:* For each $x \in R \smallsetminus \{0\}$, consider the map $R \longrightarrow R$ sending $a \mapsto ax$. Is it injective/surjective?]