Assignment 3

Fraction Fields, Polynomial Rings

1. Show that the fraction field of $\mathbb{Z}[i]$ is
$$\mathbb{Q}(i) = \{a + ib : a, b \in \mathbb{Q}\}.$$
Similarly, show that the fraction field of $\mathbb{Z}[\sqrt{2}]$ is $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

2. Let R be an integral domain. Show that $R[X]^\times = R^\times$. Can $R[X]$ be a field?

3. (a) Prove that $1 + 2X$ is a unit in $\mathbb{Z}/4\mathbb{Z}[X]$.
(*b) Determine $(\mathbb{Z}/4\mathbb{Z})[X]^\times$.
(c) Find $f \in (\mathbb{Z}/4\mathbb{Z})[X]$ of degree 2 such that $f(x) = 0$ for all $x \in \mathbb{Z}/4\mathbb{Z}$.

4. Let R be an integral domain.
(a) Prove that $R[[X]]$ is an integral domain.
(b) Prove that $1 - X \in R[[X]]^\times$.
(c) Let now $R = K$ be a field. Prove:
$$K[[X]]^\times := \left\{ \sum_{n\in\mathbb{N}} a_n X^n | a_0 \neq 0 \right\}.$$
[*Hint:* Find the coefficients of inverse power series inductively.]

5. Let R be a commutative ring.
(a) Show that there exists a unique map $D : R[X] \longrightarrow R[X]$ such that
$$D(X^i) = iX^{i-1}, \quad i \geqslant 1$$
$$D(1) = 0$$
which is R-*linear*, i.e., such that
$$\forall r \in R, \forall f, g \in R[X], \; D(rf + g) = rD(f) + D(g).$$
(b) Is D a ring homomorphism?
(c) Prove that for all $f, g \in R[X]$ one has
$$D(fg) = fD(g) + gD(f)$$

(*d) We say that $\alpha \in R$ is a *multiple root* of $f \in R[X]$ if there exists $g \in R[X]$ such that $f = (X - \alpha)^2 g$. Prove: α is a multiple root of f if and only if $f(\alpha) = D(f)(\alpha) = 0$. [*Hint:* Notice that $X^k = (X - \alpha + \alpha)^k = (X - \alpha)g_k + \alpha^k$ for some $g_k \in R[X]$ and deduce that for each $h \in R[X]$ we can write $h = (X - \alpha)\ell + h(\alpha)$ for some $\ell \in R[X]$. You'll need to use part (b) as well.]

6. Let R be a domain and $F = \mathrm{Frac}(R)$. Prove that $\mathrm{Frac}(R[X]) \cong F(X)$.