D-MATH | Algebra I | HS17
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Assignment 5

Prime and maximal ideals, Arithmetic of polynomials

1. Let R be a commutative ring. Assume that there exists an ideal $I \subset R$ such that

$$R^{\times} = R \setminus I \tag{1}$$

   (a) Show that I is a maximal ideal.

   (b) Show that I is the unique maximal ideal in R.

   (c) Conversely, assume that I is the unique maximal ideal of a commutative ring R. Prove that $R^{\times} = R \setminus I$ holds.

We call a commutative ring R *local* if there is an ideal $I = \mathfrak{m}_R \subset R$ satisfying (1) (which as just shown is equivalent to asking that $\mathfrak{m}_R$ is the unique maximal ideal of R). The field $R/\mathfrak{m}_R$ is called the *residue field* of the local ring R.

2. Let p be a prime number and consider the set

$$\mathbb{Z}_{(p)} = \left\{ x \in \mathbb{Q} : x = \frac{a}{b} \text{ for some } a, b \in \mathbb{Z},\ p \nmid b \right\}.$$

   (a) Show that $\mathbb{Z}_{(p)}$ is a commutative ring.

   (b) Show that $\mathbb{Z}_{(p)}$ is a local ring. Find its maximal ideal and its residue field.

3. Let R be a commutative ring and I, J ideals in R. Define the ideal

$$IJ := (\{ij : i \in I, j \in J\})R.$$

   (a) Why is the set $\{ij : i \in I, j \in J\}$ not necessarily an ideal?

   (b) Show that $IJ \subset I \cap J$ and find an example in which the inclusion is strict.

   (c) Prove that if I and J are coprime, then $I \cap J = IJ$.

4. (a) Consider the polynomials $p, q \in \mathbb{Q}[X]$ defined by

$$p := X^3 - \frac{5}{2}X^2 + \frac{3}{2}X \text{ and } q = 2X^2 - X - 3.$$

   Compute the Euclidean division of p by q.

   (b) Find a single generator of the principal ideal $(p, q)\mathbb{Q}[X] \subseteq \mathbb{Q}[X]$

(c) Let $K = \mathbb{C}(T)$. Compute the Euclidean division in $K[X]$ of

$$f = X^3 + TX^2 - 1 \text{ by } g = (1+T)X^2 - 1.$$

(d) Using Euclidean division in $\mathbb{F}_3[X]$, where $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ is the field of three elements, check that the ideals $(X^4 + 2X + 1)\mathbb{F}_3[X]$ and $(X^2 + X - 1)\mathbb{F}_3[X]$ are coprime.

5. Let R be a commutative ring and $I \subset R$ an ideal.

   (a) Show that for $J \subset R$ an ideal containing I, there is an isomorphism

   $$(R/I)/(J/I) \xrightarrow{\sim} R/J.$$

   (b) Show that the maximal (resp., prime) ideals of R/I are the ideals J/I where $J \subset R$ is a maximal (resp., prime) ideal containing I.

6. Find all the ideals of $\mathbb{Z}/8\mathbb{Z}$. Which are prime? Which are maximal?

7. Which of the following ideals of $\mathbb{Z}/4\mathbb{Z}[X]$ are prime? Which are maximal? [*Hint:* quotient ring]

   (a) $(X, 2)(\mathbb{Z}/4\mathbb{Z}[X]) \subset \mathbb{Z}/4\mathbb{Z}[X]$;

   (b) $2(\mathbb{Z}/4\mathbb{Z}[X]) \subset \mathbb{Z}/4\mathbb{Z}[X]$;

   (c) $(X - 1)(\mathbb{Z}/4\mathbb{Z}[X]) \subset \mathbb{Z}/4\mathbb{Z}[X]$.

8. Let R_1, R_2 be two commutative rings and $R = R_1 \times R_2$. Let $I \subset R$ be an ideal and define

   $$I_1 := \{a \in R_1 : (a, 0) \in I\} \subset R_1$$
   $$I_2 := \{b \in R_2 : (0, b) \in I\} \subset R_2.$$

   (a) Show that I_1, I_2 are ideals in R and that $I = I_1 \times I_2$.

   (b) Prove that the ideal I is maximal (resp., prime) if and only if either $I_1 = R_1$ and I_2 is maximal (resp., prime) or $I_2 = R_2$ and I_1 is maximal (resp., prime).