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Assignment 8

Normal subgroups, quotient groups, Isomorphism theorems for groups

Let G be a group. We denote by $Z(G)$ the center of G.

1. Show that $Z(S_n)$ is trivial for $n \geqslant 3$.
2. Show that any subgroup of a cyclic group is cyclic.
3. Let $G := \mathbb{R}/\mathbb{Z}$. Prove that G is isomorphic to the group $\{z \in \mathbb{C}^\times : |z| = 1\}$.
4. Let G be a group. Recall the group homomorphism $\rho : G \longrightarrow \mathrm{Aut}(G)$ seen in class, sending an element g to the automorphism $(x \mapsto gxg^{-1})$, that is the conjugation by x. We define the *group of inner automorphisms of* G as

$$\mathrm{Inn}(G) := \mathrm{Im}(\rho).$$

   (a) Prove that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

   We define the *group of outer automorphisms of* G as the quotient group $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

   (b) Determine $\mathrm{Out}(S_3)$. [*Hint:* S_3 is generated by the two permutations: $\tau : (1 \mapsto 2 \mapsto 3 \mapsto 1)$ and $\sigma_{12} : (1 \mapsto 2 \mapsto 1,\ 3 \mapsto 3)$. Use Exercise 1]

   (c) Prove that $\mathrm{Out}(\mathrm{GL}_n(\mathbb{C})) \neq \{1\}$. [*Hint:* Complex conjugation, eigenvalues]

   (d) Suppose that $\mathrm{Aut}(G)$ is cyclic. Prove: G is abelian. [*Hint:* Exercise 2]

5. Let G be a group and H, K finite subgroups of G such that $\mathrm{Card}(H)$ and $\mathrm{Card}(K)$ are coprime.

   (a) Prove that $H \cap K = \{1\}$.

   (b) Suppose moreover that G is finite and $\mathrm{Card}(G) = \mathrm{Card}(H) \cdot \mathrm{Card}(K)$. Prove that $HK = G$.

6. Let G be a group. For $a, b \in G$, define their *commutator* as

$$[a, b] := aba^{-1}b^{-1} \in G.$$

   Define the *commutator subgroup of* G as

$$[G, G] := \langle \{[a, b] : a, b \in G\} \rangle.$$

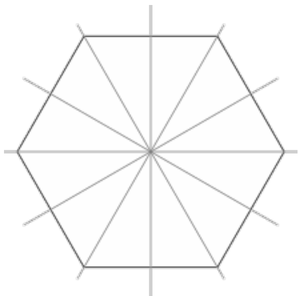
   (a) Prove that G is abelian if and only if $[G, G]$ is trivial.

(b) Prove that $[G, G] \lhd G$.

(c) The *abelianization of* G is defined as the quotient group $G^{\mathrm{ab}} := G/[G, G]$. Prove: G^{ab} is an abelian group.

(d) Let $\pi : G \longrightarrow G^{\mathrm{ab}}$ be the canonical projection. Prove: for each abelian group A and group homomorphism $\varphi : G \longrightarrow A$, there exists a unique group homomorphism $\overline{\varphi} : G^{\mathrm{ab}} \longrightarrow A$ such that $\overline{\varphi} \circ \pi = \varphi$. [*Hint:* First, show that $[G, G] \subseteq \ker(\varphi)$]

7. Let $n \geqslant 3$ be an integer. Let D_n be the group of affine transformations of $\mathbb{R}^2$ mapping a regular polygon X_n of n sides to itself. Those transformations can be described in terms of permutations of the vertices of X_n. The group D_n contains $2n$ elements: n counterclockwise rotations by $2\pi k/n$ for $k = 0, \ldots, n-1$ around the center of X_n, as well as n symmetries with respect to lines through its center. In the picture below are drown, for $n = 6$, the symmetry axes of the 6 symmetries, the 6 rotations not being represented:



(a) Let T be the rotation by $2\pi/n$, and S one of the n symmetries. Prove that $STS^{-1} = T^{-1}$ [*Hint:* Notice that an element of D_n is uniquely determined by where it maps two adjacent vertices of X_n]

(b) Notice that for each integer k the element ST^k has order 2, and deduce that

$$D_n = \{\mathrm{id}, T, \ldots, T^{n-1}, S, ST, \ldots, ST^{n-1}\}.$$

(c) Determine $Z(D_n)$ for all n.

(d) Let now $n = 4$. Prove that $\langle S \rangle \lhd \langle S, T^2 \rangle \lhd D_4$, but $\langle S \rangle \not\lhd D_4$, and determine explicitly the left and right cosets of $\langle S \rangle$ in D_4.

8. Let G be a group and H, K subgroups of G.

(a) Prove that the intersection $xH \cap yK$ of two cosets of H and K respectively is either empty or a coset of $H \cap K$.

(b) Prove that each coset of $H \cap K$ is an intersection of a coset of H with a coset of K.

(c) Prove that if H and K have finite index in G, then $H \cap K$ has finite index as well.