Prof. Emmanuel Kowalski

Assignment 14

Finitely generated modules over a PID, Elementary divisors

1. Consider the abelian group
$$A = \mathbb{Z}/250\mathbb{Z} \times \mathbb{Z}/275\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}.$$
   (a) Express A as a product of p-primary abelian groups.
   (b) Express A in terms of elementary divisors.

2. Let $m, n \in \mathbb{Z}_{>0}$ and $A = (a_{ij}) \in M_{m,n}(\mathbb{Z})$ and
$$u : \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$$
the corresponding $\mathbb{Z}$-linear map.
   (a) Show that $\mathbb{Z}^m/\mathrm{Im}(u)$ is finite if and only if A has rank m in $M_{m,n}(\mathbb{Q})$.
   (b) Suppose that $m = n$ and that $\mathbb{Z}^n/\mathrm{Im}(u)$ is finite. Prove
$$|\det(A)| = \mathrm{Card}(\mathbb{Z}^n/\mathrm{Im}(u)).$$
   (c) Let $m = n = 3$. Consider the $\mathbb{Q}$-linear map $v : \mathbb{Q}^3 \longrightarrow \mathbb{Q}^3$ whose corresponding matrix (with respect to the standard basis) is
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 4 & 1 & 1 \end{pmatrix}.$$
   Let $X = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 : 0 \leqslant x_j < 1\}$. Compute the number of points of $v(X)$ having integer coordinates [*Hint:* Let u be the map $\mathbb{Z}^3 \longrightarrow \mathbb{Z}^3$ defined by the same matrix. Define a bijection between $v(X) \cap \mathbb{Z}^3$ and $\mathbb{Z}^3/\mathrm{Im}(u)$.]

3. Let $A = \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_m\mathbb{Z}$ with $1 < d_1|d_2|\ldots|d_m$. Show that any generating set of A has $\geqslant m$ elements.

4. Let K be a field and $E = K[X]/gK[X]$, for some $g \in K[X]$ of degree $d := \deg(g) \geqslant 1$. Consider the K-linear map $u : E \longrightarrow E$ sending $[f] \mapsto [X \cdot f]$.
   (a) Compute the matrix of u in the basis $1, X, \ldots, X^{d-1}$.
   (b) Compute the characteristic polynomial of u.

5. Find the abelian group G having generators a, b, c and relations

$$\begin{aligned} -6a - 12b + 2c &= 0, \\ 7a + 8b + 7c &= 0, \\ -3a - 8b + 5c &= 0. \end{aligned}$$

[*Hint:* Work as in Example B-3.88 in J. Rotman, "Advanced modern algebra, 3rd edition, part 1".]