Solution 10

Group actions, the symmetric Group

1. Let p be a prime number and T the set of one-dimensional $\mathbb{F}_p$-subspaces in $(\mathbb{F}_p)^{n+1}$, i.e., of lines through the origin in $(\mathbb{F}_p)^{n+1}$.

   (a) Show that $\mathrm{GL}_{n+1}(\mathbb{F}_p)$ acts transitively on T by $g \cdot L = g(L)$.

   (b) Compute the stabilizer of the line $L_0 := \langle (1, 0, \ldots, 0) \rangle \in T$.

   (c) Compute $\mathrm{Card}(T)$. [*Hint:* T has the same number of elements of the set of orbits of $\mathbb{F}_p^\times$ acting on $(\mathbb{F}_p)^{n+1} \smallsetminus \{0\}$]

   Solution:

   (a) In the following, all 'lines' are supposed to be line through the origin.
   Under the identification $\mathrm{GL}_{n+1}(\mathbb{F}_p) \cong \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_p^{n+1})$, in the statement $g \cdot L$ is defined as the image of the line L under g, which we know is again a line. The axioms of group action are trivially checked.
   A line in $\mathrm{GL}_{n+1}(\mathbb{F}_p)$ can be determined by a non-zero vector. Since any non-zero vector v_0 of $\mathbb{F}_p^{n+1}$ can be completed to a basis, there exists for each $v_0 \neq 0$ an automorphism of the $\mathbb{F}_p$-vector space $\mathbb{F}_p^{n+1}$ sending $(1, 0, \ldots, 0) \mapsto v_0$ (in other words, the action of $\mathrm{GL}_{n+1}(\mathbb{F}_p)$ on $\mathbb{F}_p^{n+1}$ is transitive). Such an automorphism sends the line L_0 generated by $(1, 0, \ldots, 0)$ to the one generated by v_0, which is a arbitrary. Hence the $\mathrm{GL}_{n+1}(\mathbb{F}_p)$-orbit of L_0 is T and the $\mathrm{GL}_{n+1}(\mathbb{F}_p)$-action on T is transitive.

   (b) An automorphism of $\mathbb{F}_p^{n+1}$ sends $L_0 \mapsto L_0$ if and only if it sends $(1, 0, \ldots, 0)$ to $(\lambda, 0, \ldots, 0)$ for $\lambda \in \mathbb{F}_p$. Hence

   $$\mathrm{Stab}(L_0) = \left\{ \begin{pmatrix} * & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \ldots & * \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{F}_p) \right\}.$$

   (c) One can use the hint to obtain that

   $$\mathrm{Card}(T) = \frac{p^{n+1} - 1}{p - 1} = 1 + p + \cdots + p^n.$$

   Alternatively, one can use the orbit stabilizer theorem, which tells us that

   $$T \cong \mathrm{GL}_{n+1}(\mathbb{F}_p)/\mathrm{Stab}(L_0).$$

We notice that $\mathrm{Card}(\mathrm{Stab}(L_0)) = (p-1)p^n \mathrm{Card}(\mathrm{GL}_n(\mathbb{F}_p))$, since the upper left entry can be any element of $\mathbb{F}_p^\times$, the remaining n elements in the first row are arbitrary elements of $\mathbb{F}_p$ and the remaining element for an invertible $n \times n$ matrix. For the cardinality of $\mathrm{GL}_n(\mathbb{F}_p)$, one can generalize the counting of bases in Assignment 7, Exercise 3 and obtain

$$\mathrm{Card}(\mathrm{GL}_n(\mathbb{F}_p)) = (p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}).$$

Altogether, this lets us compute

$$\begin{aligned}\mathrm{Card}(T) &= \frac{\mathrm{Card}(\mathrm{GL}_{n+1}(\mathbb{F}_p))}{(p-1)p^n\mathrm{Card}(\mathrm{GL}_n(\mathbb{F}_p))} \\ &= \frac{(p^{n+1}-1)(p^{n+1}-p)\cdots(p^{n+1}-p^n)}{(p-1)p^n(p^n-1)(p^n-p)\cdots(p^n-p^{n-1})} = \frac{p^{n+1}-1}{p-1}.\end{aligned}$$

2. Consider the standard action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$. Determine the orbits of $(1,0)$ under each of the subgroups

$$H_1 := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\},\ H_2 := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\},\ H_3 := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\},\ H_4 := \mathrm{SO}_2(\mathbb{R}).$$

Solution: This is done by looking at the images of $(1,0)$ under the matrices of the given forms, which coincides with the first column of the matrix.

- Since the first column of $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for all t, the H_1-orbit of $(1,0)$ is $\{(1,0)\}$.
- The matrices in H_2 and H_3 have all first column equal to $\begin{pmatrix} a \\ 0 \end{pmatrix}$ for $a \in \mathbb{R}^\times$. Hence the H_2-orbit and the H_3-orbit of $(1,0)$ coincide and they are given by $\{(a,0) :\}$, the horizontal line through the origin, removed of the origin.
- Since a generic element of $\mathrm{SO}_2(\mathbb{R})$ is a matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta \in \mathbb{R}$, we can conclude that the H_4-orbit of $(1,0)$ is $\{(\cos\theta, \sin\theta) \in \mathbb{R}^2\} = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$.

3. Let G be a group acting on a set T. Fix $x_0 \in T$. Let $H \subset G$ be a subgroup and define X to be the H-orbit of x_0. Show that, for $g \in G$,

$$g \cdot X = \{g \cdot x : x \in X\}$$

is the gHg^{-1}-orbit of $g \cdot x_0$.

Solution: The gHg^{-1}-orbit of $g \cdot x_0$ is

$$(gHg^{-1}) \cdot (g \cdot x_0) = \{(ghg^{-1}) \cdot (g \cdot x_0) : h \in H\} = \{g \cdot (h \cdot x_0) : h \in H\} = g \cdot X,$$

where the last equality is due to the fact that the H-orbit of x_0 is, by definition, the set of elements $h \cdot x_0$ with $h \in H$.

4. Let $\sigma \in S_n$. Denote by $F(\sigma)$ the number of points fixed by σ. Prove that the following formulas hold:

$$\frac{1}{n!}\sum_{\sigma\in S_n} F(\sigma) = 1$$
$$\frac{1}{n!}\sum_{\sigma\in S_n} F(\sigma)^2 = 2$$

[*Hint:* Notice that $F(\sigma) = \sum_{x:\sigma(x)=x} 1$. Invert the order of summation.]

Solution: As suggested by the hint, we see that $F(\sigma) = \sum_{x:\sigma(x)=x} 1$. Then

$$\frac{1}{n!}\sum_{\sigma\in S_n} F(\sigma) = \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{x:\sigma(x)=x} 1 = \frac{1}{n!}\sum_{x=1}^{n}\sum_{\sigma\in S_n:\sigma(x)=x} 1 = \frac{1}{n!}\sum_{x=1}^{n}(n-1)! = \frac{1}{n}\cdot n = 1.$$

We have used the fact that $\mathrm{Card}(\mathrm{Stab}_{S_n}(x)) = (n-1)!$, since a permutation $\sigma \in S_n$ fixing x is given an arbitrary partition of $\{1,\dots,n\}\setminus\{x\}$. For the average of $(F(\sigma))^2$, we first look at the product of sums

$$(F(\sigma))^2 = \Big(\sum_{x:\sigma(x)=x} 1\Big)\Big(\sum_{y:\sigma(x)=x} 1\Big) = \sum_{\substack{x,y\in\{1,\dots,n\}\\ \sigma(x)=x\\ \sigma(y)=y}} 1.$$

Then, switching the summation order,

$$\begin{aligned}\frac{1}{n!}\sum_{\sigma\in S_n}(F(\sigma))^2 &= \frac{1}{n!}\sum_{\sigma\in S_n}\sum_{\substack{x,y\in\{1,\dots,n\}\\ \sigma(x)=x\\ \sigma(y)=y}} 1 = \frac{1}{n!}\sum_{x,y\in\{1,\dots,n\}}\sum_{\substack{\sigma\in S_n\\ \sigma(x)=x\\ \sigma(y)=y}} 1\\ &= \frac{1}{n!}\Big(\sum_{\substack{x,y\in\{1,\dots,n\}\\ x\neq y}}\sum_{\substack{\sigma\in S_n\\ \sigma(x)=x\\ \sigma(y)=y}} 1 + \sum_{x\in\{1,\dots,n\}}\sum_{\substack{\sigma\in S_n\\ \sigma(x)=x}} 1\Big) =\\ &= \frac{1}{n!}\Big(n\cdot(n-1)\cdot(n-2)! + n\cdot(n-1)!\Big) = 2\end{aligned}$$

5. For each conjugacy class S_6, write down a representative and the cardinality of the class.

   Solution: As seen in class, the elements in a conjugacy class of S_6 are all those with a specific *cyclic type*, that is, a specific length of the cycles appearing in their unique decomposition into disjoint cycles, where we also include 1-cycles in order to obtain an unordered partition of n. Hence the conjugacy classes of S_6 are given by partitions of 6, which we list in an order for which their number of summands increases. In a single partition the order of the summands is irrelevand, so we sum them in decreasing order. This gives:

- $6 = 6$, the trivial partition. We obtain the class of 6-cycles, $[(1\,2\,3\,4\,5\,6)]$. The number of elements is the number of 6-cycles in S_6. Notice that such a 6-cycle can always be written as $(1\,a_1\,a_2\,a_3\,a_4\,a_5)$, and that different choices of $(a_1, a_2, a_3, a_4, a_5)$ lead to different 6-cycles. Hence there are $5! = 120$ elements in this class.
- $6 = 5 + 1$. This corresponds to the class of 5-cycles, $[(1\,2\,3\,4\,5)]$. The number of 5-cycles in S_6 can be easily determined. There are 6 subsets of 5 elements in $\{1, \ldots, 6\}$. For each of those, we can write any 5-cycle with first element equal to the minimal one, and freely choose the remaining 4 elements. Hence there are $6 \cdot 4! = 144$ elements in this class.
- $6 = 4+2$. This corresponds to the class $[(1\,2)(3\,4\,5\,6)]$. In order to determine how many elements with this cycle type are there, we first notice that there are $\binom{6}{2} = 15$ ways to choose 2 elements in $\{1, \ldots, 6\}$. For each of those, the 2-cycle is uniquely determined, while the 4-cycle can be chosen with first element equal to the minimum of the four elements involved, and the remaining 3 elements freely chosen, giving 3! possibilities. Hence this class contains $15 \cdot 3! = 90$ elements.
- $6 = 3+3$. This corresponds to the class $[(1\,2\,3)(4\,5\,6)]$. In order to determine how many elements with this cycle type are there, we first notice that there are $\frac{1}{2!}\binom{6}{3} = 10$ ways to choose 2 disjoint triples of elements (without caring about the order of the two couples) in $\{1, \ldots, 6\}$. The two subsets of three elements can be independently put in one of the 2 distinct three cycles, so that we obtain $10 \cdot 2 \cdot 2 = 40$ elements in this conjugacy class.
- $6 = 4 + 1 + 1$. This corresponds to the class of 4-cycles, $[(1\,2\,3\,4)]$. There are $\binom{6}{4} = 15$ subsets of 4 elements in $\{1, \ldots, 6\}$. For each of those, as already seen above, there are 3! possible 4-cycles. Hence this class contains $15 \cdot 6 = 90$ elements.
- $6 = 3 + 2 + 1$. This corresponds to the class $[(1\,2\,3)(4\,5)]$. In order to determine how many elements with this cycle type are there, we first notice that there are $\binom{6}{3}\binom{3}{2} = 60$ ways to choose disjoint subsets of 3 and 2 elements in $\{1, \ldots, 6\}$. For each of those, the 2-cycle is determined, while for the 3-cycle there are 2 possibilities. Hence we have $60 \cdot 2 = 120$ elements in this conjugacy class.
- $6 = 2 + 2 + 2$. This corresponds to the class $[(1\,2)(3\,4)(5\,6)]$. In order to determine how many elements with this cycle type are there, we first notice that there are $\frac{1}{3!}\binom{6}{2}\binom{4}{2} = 15$ ways to choose 3 disjoint couple of elements in $\{1, \ldots, 6\}$ (without caring of the order of the couples). For each of those, the 2-cycles are uniquely determined, so that we have 15 elements in this conjugacy class.
- $6 = 3 + 1 + 1 + 1$. This corresponds to the class of 3-cycles, $[(1\,2\,3)]$. There are $\binom{6}{3} = 20$ subsets of 3 elements in $\{1, \ldots, 6\}$. For each of those, as already

seen above, there are 2 possible 3-cycles. Hence this class contains $20 \cdot 2 = 40$ elements.

- $6 = 2 + 2 + 1 + 1$. This corresponds to the class $[(1\,2)(3\,4)]$. In order to determine how many elements with this cycle type are there, we first notice that there are $\frac{1}{2!}\binom{6}{2}\binom{4}{2} = 45$ ways to choose two disjoint couple of elements in $\{1, \ldots, 6\}$ (without caring of the order of the couples). For each of those, the 2-cycles are uniquely determined, so that we have 45 elements in this conjugacy class.
- $6 = 2+1+1+1+1$. This corresponds to the class of 2-cycles, $[(1\,2)]$. There are $\binom{6}{2} = 15$ subsets of 2 elements in $\{1, \ldots, 6\}$. For each of those, the 2-cycle is already determined. Hence this class contains 15 elements.
- $6 = 1+1+1+1+1+1$. This correspond to the class [id], which consist of 1 element.

We hence found the following conjugacy classes:

partition	representative	cardinality
$6 = 6$	$(1\,2\,3\,4\,5\,6)$	120
$6 = 5 + 1$	$(1\,2\,3\,4\,5)$	144
$6 = 4 + 2$	$(1\,2)(3\,4\,5\,6)$	90
$6 = 3 + 3$	$(1\,2\,3)(4\,5\,6)$	40
$6 = 4 + 1 + 1$	$(1\,2\,3\,4)$	90
$6 = 3 + 2 + 1$	$(1\,2\,3)(4\,5)$	120
$6 = 2 + 2 + 2$	$(1\,2)(3\,4)(5\,6)$	15
$6 = 3 + 1 + 1 + 1$	$(1\,2\,3)$	40
$6 = 2 + 2 + 1 + 1$	$(1\,2)(3\,4)$	45
$6 = 2 + 1 + 1 + 1 + 1$	$(1\,2)$	15
$6 = 1 + 1 + 1 + 1 + 1 + 1$	id	1

Aliter: In order to compute the cardinalities, one could use the general formula obtained in Exercise 8d), which says that the cardinality of the conjugacy class associated to the partition $6 = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_6 \cdot 6$ is $6!/(\prod_{i=1}^{6} k_i! i^{k_i})$. For instance,

$$\mathrm{Card}[(1\,2\,3)(4\,5\,6)] = \frac{6!}{2!3^2} = 4 \cdot 5 \cdot 2 = 40, \text{ since } k_i = \begin{cases} 2 & i = 3 \\ 0 & i \neq 3 \end{cases}$$

6. Let $n \geqslant 3$. Prove that $[S_n, S_n] = A_n$. [Recall: for a group G, the commutator $[G, G]$ is defined as the subgroup of G generated by $\{aba^{-1}b^{-1} : a, b \in G\}$. See Assignment 8, Exercise 6]

   Solution: Consider the signature morphism $\varepsilon : S_n \longrightarrow \{\pm 1\}$. Since $\{\pm 1\}$ is an abelian group, we have $[S_n, S_n] \subset \ker(\varepsilon) = A_n$ as seen in Assignment 8, Exercise 6(d).

Conversely, in order to prove that $A_n \subset [S_n, S_n]$, it is enough to check that $[S_n, S_n]$ contains all 3-cycles, as those are generators of A_n. This is indeed the case, because, as Exercise 6 suggests,

$$(a\,b)(b\,c) = (a\,b\,c),$$

and by the conjugation rules seen in class

$$(b\,c) = (a\,c)(b\,a)(a\,c)^{-1},$$

so that

$$(a\,b\,c) = (a\,b)(a\,c)(b\,a)(a\,c) = (a\,b)(b\,c)(a\,b)(b\,c) = [(a\,b), (b\,c)] \in [S_n, S_n].$$

Aliter: The second inclusion can be proven in another way for $n \geqslant 5$. Indeed, in this case A_n is simple, so on can observe that $1 \neq [(1\,2), (2\,3)] \in [S_n, S_n] \lhd S_n$, so that $\{1\} \neq [S_n, S_n] \lhd A_n$ and hence $[S_n, S_n] = A_n$.

7. (a) Prove that S_n is generated by $\{\sigma_i := (i \quad i+1),\ 1 \leqslant i \leqslant n-1\}$, and that those generators satisfy the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j| \geqslant 2$$
$$(\sigma_i\sigma_{i+1})^3 = \text{id, for } 1 \leqslant i \leqslant n-2.$$

(b) Let $\tau := (1\,2\,\ldots\,n)$. Show that S_n is generated by $\{\sigma_1, \tau\}$. [*Hint:* Express σ_i in terms of σ_1 and τ]

Solution:

(a) First, notice that S_n is generated by $\{(1\ i) : 2 \leqslant i \leqslant n-1\}$. Indeed, for each $i, j \in \{1, \ldots, j\} \smallsetminus \{1\}$, there is an equality

$$(1\,i)(1\,j)(1\,i) = (i\,j),$$

so that $\{(1\ i) : 2 \leqslant i \leqslant n-1\}$ generates all transpositions, which is enough to generate S_n as seen in class. In order to conclude, it is enough to check for each $2 \leqslant i \leqslant n-1$ that $(1\ i) \in \langle\{\sigma_k : 1 \leqslant k \leqslant n-1\}\rangle$. We do it by induction on i. This is clear for $i = 2$, since $(1\,2) = \sigma_1$. Suppose that $(1\ i-1) \in \langle\{\sigma_k : 1 \leqslant k \leqslant n-1\}\rangle$. Then

$$(1\ i) = \sigma_{i-1}(1\ i-1)\sigma_{i-1}^{-1} \in \langle\{\sigma_k : 1 \leqslant k \leqslant n-1\}\rangle,$$

which concludes the proof that the elements σ_i generate S_3.
We observe that σ_i and σ_j are disjoint when $|i-j| > 2$, so that they commute, while $\sigma_i\sigma_{i+1} = (i\ i+1\ i+2)$ is a 3-cycle and, as such, it has order 3.

(b) By the conjugation formula seen in class, for each $1 \leqslant k \leqslant n-1$,

$$\tau^{k-1}(1\,2)\tau^{-k+1} = (k\,k+1) = \sigma_k,$$

so that $\langle\{\sigma_1, \tau\}\rangle$ contains all the elements σ_k and by part (a) it coincides with S_3.

8. Let $n \geqslant 2$ be an integer and $k_i \in \mathbb{Z}_{>0}$ for $i = 1, \ldots, n$ be such that

$$k_1 \cdot 1 + k_2 \cdot 2 + \ldots + k_n \cdot n = n.$$

Let X be the conjugacy class of X determined by $(k_1, \ldots, k_n)$. Tautologically, S_n acts on X by conjugation and the action is transitive.

(a) Fix $\sigma_0 \in X$ and let $H = \mathrm{Stab}_{S_n}(\sigma_0)$. Prove that

$$\mathrm{Card}(H) = \prod_{i=1}^{n} i^{k_i} \cdot k_i!$$

(b) Use the above to write down an expression for Card(X).

(c) Show that Card$(\{n -$ cycles in $S_n\}) = (n-1)!$

Solution:

(a) As seen in class, a conjugation class in S_n is uniquely determined by an unordered partition of n, that is, a way of writing n as a sum of positive integers, which gives the cycle type of any elements in the considered conjugation class in S_n [For instance, $3+2=5$ corresponds to the conjugation class consisting of all permutations $(a_1\ a_2\ a_3)(a_4\ a_5) \in S_5$ for distinct a_j]. In the notation of the exercise, k_i is the number of times that the number $i \in \{1, \ldots, n\}$ appears in the partition of n. Hence the class X consist of the permutations of the form

$$(a_{1,1,1})\cdots(a_{1,1,k_1})(a_{2,1,1}\ a_{2,2,1})\cdots(a_{2,1,k_2}\ a_{2,2,k_2})\cdots(a_{n,1,1}\ \cdots\ a_{n,n,1})\cdots(a_{n,1,1}\ \cdots\ a_{n,n,1})$$

where the $a_{\lambda,\mu,\nu}$ all distinct and $\{a_{\lambda,\mu,\nu}\} = \{1, \ldots, n\}$. As visible in the above expression, λ stands for the length of the cycle, μ stands for the position in the cycle expression and ν for the position of the cycle among those of same length.

Call σ_0 the element written in the above expression in terms of $a_{\lambda,\mu,\nu}$. As seen in class, for $\sigma \in S_n$, the conjugate $\sigma\sigma_0\sigma^{-1}$ can be written with the same expression, by replacing each $a_{\lambda,\mu,\nu}$ with $\sigma(a_{\lambda,\mu,\nu})$.

Since the $a_{\lambda,\mu,\nu}$ are distinct, so are the $\sigma(a_{\lambda,\mu,\nu})$ and, as the decomposition into disjoint cycles is unique up to order, we obtain that $\sigma\sigma_0\sigma^{-1} = \sigma_0$ if and only if the cycles of the same length are permuted, and the elements appearing in the same cycle are permuted in a cyclic way (since $(a_1\, a_2\, \cdots\, a_m) = (b_1\, b_2\, \cdots\, b_m)$ if and only if there exists $t \in \mathbb{Z}$ such that $b_j \equiv a_j + t \pmod{m}$). In terms of the indices, this means that

$$\sigma(a_{\lambda,\mu,\nu}) = a_{\lambda,\mu+t_{\lambda,\nu},\nu'_\lambda}, \tag{1}$$

where for each λ the index ν' is independent on μ (which gives $k_\lambda!$ choices) and $t_{\lambda,\nu} \in \{0, \ldots, \lambda - 1\}$ for independently for each ν (which gives λ^{k_λ} choices). [Notice that the index $\mu + t_{\lambda,\nu}$ in (1) needs to be interpreted modulo λ.] Hence, for each λ, we have $k_\lambda! \cdot \lambda^{k_\lambda}$ choices, so that

$$\mathrm{Card}(H) = \prod_{\lambda=1}^{n} k_\lambda! \cdot \lambda^{k_\lambda}$$

as desired.

(b) By the orbit stabilizer theorem, since the action on S_n is transitive, we obtain

$$\mathrm{Card}(X) = \frac{\mathrm{Card}(S_n)}{\mathrm{Card}(H)} = \frac{n!}{\prod_{\lambda=1}^{n} k_\lambda! \cdot \lambda^{k_\lambda}}.$$

(c) The n-cycles in S_n form the conjugacy class corresponding to the trivial partition $n = n$, i.e., $k_n = 1$ and $k_j = 0$ for $j < n$. We can apply the previous part in this special case to obtain:

$$\mathrm{Card}(\{n - \text{cycles in } S_n\}) = \frac{n!}{\sum_{\lambda=1}^{n} k_\lambda! \cdot \lambda^{k_\lambda}} = \frac{n!}{0 + \ldots + 0 + 1! \cdot n^1} = (n-1)!$$