Classical Modular Forms Lecture 1, Sept. 18, 2017.

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The Zeta function: (Riemann)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For every $\delta > 1$, converges absolutely and uniformly on $\Re s \geq \delta > 1$.

First introduced by Euler (Swiss) 1737 to study distribution of primes.

Fundamental Theorem of Arithmetic:

All positive integers factor uniquely into a product of prime numbers.

Euclid: There are infinitely many primes.

Euler: FTA $\Rightarrow$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(\sum_{a \geq 0} \frac{1}{p^{as}} \right) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Euler's analytic proof of infinitude of primes:

If # of primes were finite, $\prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$ would be a finite product and therefore ~~converge~~ takes a finite value @ $s=1$

On the other hand: $\lim_{s \to 1^+} \sum_{n=1}^{\infty} \frac{1}{n^s} = +\infty$ (compare w/ integral)

Why should we care? This proof shows $\sum_{p \text{ prime}} \frac{1}{p} = +\infty$ as well (see Exercises) which says something about the density of primes, since if the primes were very sparse in $\mathbb{N}$, the sum would converge.

Riemann 1859: Complexify the variable s.

(Note: $\frac{dt}{t}$ is the Haar measure for $(\mathbb{R}_+^*, \times)$)

Let $\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$ the Gamma function.

Converges absolutely and uniformly ~~for all~~ on $Re(s) \geq \delta$ for all $\delta > 0$.

$\Gamma(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{0,-1,-2,\ldots\}$ with simple poles at $0,-1,-2,\ldots$ and no zeros. (Exercises).

(we more or less understand $\Gamma(s)$ perfectly)

Let $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

<u>Theorem</u> (Riemann)

The function $\xi(s)$ admits an analytic continuation to $\mathbb{C}$ with simple poles only at $s = 0, 1$. Moreover,

$$\xi(s) = \xi(1-s).$$

Riemann Hypothesis: All zeros of $\xi(s)$ have $Re(s) = 1/2$.

(Note: We know $\approx 41\%$ of zeros satisfy RH, and all zeros ρ with $|Im(\rho)| \leq 10^{13}$ satisfy RH).

<u>Proof (of Theorem):</u>

Express $\xi(s)$ as a Mellin transform:

$$\xi(s) = \int_0^\infty \left(\sum_{n \geq 1} e^{-\pi n^2 t} \right) t^{s/2} \frac{dt}{t} \quad \text{for } Re(s) > 1.$$

Indeed: $\pi^{-s/2} \Gamma(s/2) \frac{1}{n^s} = \int_0^\infty e^{-t} \left(\frac{t}{n^2 \pi} \right)^{s/2} \frac{dt}{t} = \int_0^\infty e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}$.

Sum over $n \geq 1 \Rightarrow$ above formula

Let $\omega(t) = \sum_{n \geq 1} e^{-\pi n^2 t}$, so $\xi(s) = \int_0^\infty \omega(t) t^{s/2} dt/t$.

We want to show that $\xi(s)$ converges for all $s \neq 0, 1$.

Note: $\omega(t) \approx e^{-\pi t}$ as $t \to \infty$, so convergence at $t = \infty$ is OK.

Idea: Split $\int$ at $t = 1$, and change variables $t \leftrightarrow 1/t$

$$\xi(s) = \int_0^\infty \omega(t) t^{s/2} \frac{dt}{t} = \int_1^\infty \omega(t) t^{s/2} \frac{dt}{t} + \int_1^\infty \omega(1/t) t^{-s/2} \frac{dt}{t}.$$

Let $\Theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = 1 + 2\omega(t)$.

or better, let $f_t(x) = e^{-\pi t x^2}$, then $\Theta(t) = \sum_{n \in \mathbb{Z}} f_t(n)$

Now take advantage of the fact that $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$.

<u>Theorem</u> (Poisson Summation)

Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class of fcns on $\mathbb{R}$.

One has for $u \in \mathbb{R}$

$$\sum_{n \in \mathbb{Z}} f(n+u) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e(-nu)$$

where $\hat{f}(y) = \int_{\mathbb{R}} f(x) e(-xy)\, dx$ is the Fourier transform

Note: we follow a standard analytic number theory convention and write $e(x) = e^{2\pi i x}$.

Recall the Gaussian is self-dual: $\widehat{e^{-\pi x^2}}(y) = e^{-\pi y^2}$, so by a change of variables: $\hat{f}_t(y) = t^{-1/2} f_{1/t}(y)$.

$$\Rightarrow \boxed{\Theta(1/t) = t^{1/2} \Theta(t)}$$

So: $\xi(s) = \int_0^\infty \omega(t) t^{s/2} \frac{dt}{t} = \int_1^\infty \frac{\Theta(t) - 1}{2} t^{s/2} \frac{dt}{t} + \int_1 \frac{t^{1/2}\Theta(t) - 1}{2} t^{-s/2} \frac{dt}{t}$

$= \int_1^\infty \frac{\Theta(t) - 1}{2} t^{s/2} + \int_1^\infty \frac{\Theta(t) - 1}{2} t^{\frac{1-s}{2}} \frac{dt}{t} + \frac{1}{s} + \frac{1}{1-s}.$ Q.E.D.

We follow Riemann's lead and <u>complexify</u>.

Let $it = z$, let $\tilde{\Theta}(z) = \Theta(t) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} = \sum_{n\in\mathbb{Z}} e(n^2 z/2)$

Converges absolutely and uniformly on $\operatorname{Im} z \geq \delta \ \forall \delta > 0$.

So defines a holomorphic function on

$$\mathcal{H} := \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}.$$

We take $\sqrt{z} = \exp(\tfrac{1}{2}\log z)$, which defines $\sqrt{z}$ on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, so has usual value on $\mathbb{R}_{>0}$.

Similarly to before: $\tilde{\Theta}\left(\frac{-1}{z}\right) = \sqrt{-iz}\,\tilde{\Theta}(z)$ and

$$\tilde{\Theta}(z+2) = \tilde{\Theta}(z).$$

We interpret these as <u>Möbius transformations</u> on $\mathcal{H}$.

Let $GL_2^+(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2\mathbb{R} : \det g > 0 \right\}$.

$GL_2^+\mathbb{R} \curvearrowright \mathcal{H}$ by fractional linear transformations:

$$g.z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Check: $\operatorname{Im}(g.z) = \det g . \dfrac{\operatorname{Im} z}{|cz+d|^2} > 0$, so $\mathcal{H}$ is preserved.

These are <u>Holomorphic</u> transformations.

Note:

Scalars $\left\{ \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\} \subseteq GL_2^+(\mathbb{R})$ act trivially: $\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} z = z$.

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $z \mapsto z+2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} z = T^2 z$

So $\tilde{\Theta}(T^2 z) = \tilde{\Theta}(z)$.

Let $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, so $-1/z = Sz$, and $\tilde{\Theta}(Sz) = \sqrt{-iz}\,\tilde{\Theta}(z)$.

So $\tilde{\Theta}(z)$ satisfies a transformation law for any

$\gamma \in \langle S, T^2 \rangle \subseteq SL_2\mathbb{Z} \subseteq GL_2^+\mathbb{R}$.

More explicity, we show $\tilde{\Theta}(z)$ transforms under $\Gamma(2) \subseteq SL_2\mathbb{Z}$

$\Gamma(2) := \{\gamma \in SL_2\mathbb{Z} : b \equiv c \equiv 0\ (2)\} = \ker\left(SL_2\mathbb{Z} \to SL_2\mathbb{Z}/2\right)$ reduce mod 2.

So a normal subgroup of $SL_2\mathbb{Z}$.

Split into two cases: $c=0$, $c \neq 0$.

Case ① $c=0$ Then $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} \pm 1 & 2b' \\ & \pm 1 \end{pmatrix}$ with $b = 2b'$, since $ad-bc=1$, Either both + or both −

In this case $\gamma = \pm T^{2b'}$, so $\tilde{\Theta}(\pm T^{2b'} z) = \tilde{\Theta}(z)$.

Case ② $c \neq 0$. Since $-Id$ acts trivially, assume $c>0$.

Then $\gamma z = \dfrac{az+b}{cz+d} = \dfrac{a}{c} - \dfrac{1}{c(cz+d)}$, so

$$\tilde{\Theta}(\gamma z) = \sum_{n\in\mathbb{Z}} \underbrace{e\left(\frac{n^2 a}{2c}\right)} \exp\left(\frac{-\pi i n^2}{c(cz+d)}\right)$$

Note: this only depends on $n \bmod c$.

Indeed, c is even, and if $n = \alpha + mc$, $m \in \mathbb{Z}$ then
$n^2 = \alpha^2 + 2\alpha mc + m^2c^2 \equiv \alpha^2 \bmod c$.

So $$\tilde{\Theta}(\gamma z) = \sum_{\alpha \bmod c} e\left(\frac{a\alpha^2}{2c}\right) \sum_{m\in\mathbb{Z}} \exp\left(\frac{-\pi i c(\alpha/c + m)^2}{cz+d}\right)$$

Apply Poisson Summation for f_t w/ $t = \dfrac{ic}{cz+d}$, $u = \alpha/c$

$\Rightarrow$

$$\tilde{\Theta}(\gamma z) = \sum_{\alpha \bmod c} e\left(\frac{a\alpha^2}{2c}\right) \left(\frac{cz+d}{ic}\right)^{1/2} \sum_{m\in\mathbb{Z}} \exp\left(i\pi m^2 \frac{cz+d}{c}\right) e\left(\frac{-m\alpha}{c}\right)$$

$$= \left(\frac{cz+d}{ci}\right)^{1/2} \sum_{m\in\mathbb{Z}} e\left(\frac{m^2 z}{2}\right) \sum_{\alpha(c)} e\left(\frac{\alpha^2 a - 2m\alpha + m^2 d}{2c}\right)$$

$$= \frac{1}{2}\left(\frac{cz+d}{ci}\right)^{1/2} \sum_{m\in\mathbb{Z}} e\left(\frac{m^2 z}{2}\right) \sum_{\alpha(2c)} e\left(\frac{\alpha^2 a + 2m\alpha + dm^2}{2c}\right).$$

Note: $ad \equiv 1 \bmod 2c$ since b is even. So let $\bar{a} \equiv$ inverse of $a \bmod 2c$ $\equiv d \ (2c)$.

So $a\alpha^2 + 2\alpha m + dm^2 \equiv a(\alpha + \bar{a}m)^2 \quad (2c)$

Change variables $\alpha \mapsto \alpha + \bar{a}m \bmod 2c$

$$\sum_{\alpha(2c)} e\left(\frac{\alpha^2 a + 2m\alpha + dm^2}{2c}\right) = \underbrace{\sum_{\alpha(2c)} e\left(\frac{a\alpha^2}{2c}\right)}_{\text{a Gauss sum}} =: G(a,2c)$$

if $(a,2c)=1$.

So: $\tilde{\Theta}(\gamma z) = \frac{1}{2} G(a,2c)\left(\frac{cz+d}{ci}\right)^{1/2} \tilde{\Theta}(z)$.

Note: If $(b,c)=1$ are coprime, then $G(ab^2,c) = G(a,c)$ by changing variables. In particular, if $ad \equiv 1 \ (c)$ then $G(a,c) = G(ad^2,c) = G(d,c)$.

So $$\boxed{\tilde{\Theta}(\gamma z) = \frac{G(d,c)}{2(ic)^{1/2}} (cz+d)^{1/2} \tilde{\Theta}(z)}$$

Alternate calculation: Let $w = Sz$ (change variables), note $S^2 = -Id$, so $\tilde{\Theta}(\gamma z) = \tilde{\Theta}(\gamma S w)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} =: \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \gamma'$

Recall $\tilde{\Theta}(-1/z) = \sqrt{iz}\,\tilde{\Theta}(z)$, so $\tilde{\Theta}(w) = \sqrt{-iz}\,\tilde{\Theta}(z)$.

So to compute $\tilde{\Theta}(\gamma z)$ it suffices to compute $\tilde{\Theta}(\gamma' w) = \tilde{\Theta}(\gamma z)$.
Note γ' has $a' \equiv d' \equiv 0 \ (2)$ and $b', c' \equiv 1 \ (2)$.

A similar calculation shows:

Note: $\frac{a'}{2}\alpha^2 - m\alpha + \frac{d'}{2}m^2 \equiv \frac{a'}{2}(\alpha^2 - 2\bar{a}m\alpha + \bar{a}d'm^2) \bmod c$

$$\tilde{\Theta}(\gamma' w) = G(a'/2, c')\left(\frac{c'w+d'}{ic'}\right)^{1/2} \tilde{\Theta}(w)$$
$$= G(-c/2, d)\left(\frac{cz+d}{d}\right)^{1/2} \tilde{\Theta}(z),$$ so we have shown

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<u>Theorem</u> If $c,d>0$, coprime and c even, then
$$\frac{G(-2c;d)}{d^{1/2}} = \frac{G(d,2c)}{2(ic)^{1/2}}$$