[HAPPY HALOWEEN]

Summary of last time:

Let $\Gamma \subseteq SL_2\mathbb{Z}$ finite index subgroup.

Theorem 1 The space of orbits $Y(\Gamma) = {}_{\Gamma}\backslash\mathcal{H}$ under the quotient topology is a connected, locally compact, Hausdorff space.

Theorem 2 Let $\Gamma \subseteq SL_2\mathbb{Z}$ a finite-index subgroup. For any $z, z' \in \mathcal{H}$ one has

① For any open sets U_1, U_2 with compact closures, the set
$$\{\gamma \in \Gamma : \gamma U_1 \cap U_2 \neq \emptyset\} \text{ is finite.}$$

② If $z_1 \notin \Gamma z_2$ then $\exists\ U_1 \ni z_1,\ U_2 \ni z_2$ such that
$$\Gamma(U_1) \cap U_2 = \emptyset.$$

③ There exists $U \ni z$ such that
$$\{\gamma \in \Gamma : \gamma U \cap U \neq \emptyset\} = \Gamma_z$$
$\uparrow$ stabilizer of z.

④ The stabilizer of z, Γ_z is finite.

Pf ① Showed in the course of the proof of the prop from last time.

② Since $\gamma z_1 \neq z_2$ $\forall \gamma \in \Gamma$ we have $\Gamma(U_1) \cap U_2 = \emptyset$ by the proposition

③ This is the proposition applied to $z_1 = z_2 = z$.

④ ~~Apply proposition to $z_1 = z_2 = z$~~ Γ is discrete in $SL_2\mathbb{R}$ and $SL_2\mathbb{R}_z$ is compact, being conjugate to $SO_2(\mathbb{R})$. So $(SL_2\mathbb{R})_z \cap \Gamma$ is finite.

"Elliptic Points"

Def For $\Gamma \subseteq SL_2\mathbb{Z}$, $z \in \mathcal{H}$ is called an elliptic point if $\Gamma_z \neq \{\pm Id\}$. $\pi(z) \in Y(\Gamma)$ is also called an elliptic point then.

Prop For any $z \in \mathcal{H}$, $(SL_2\mathbb{Z})_z$ is a finite cyclic group of order 2, 4, 6. The second and third possibilities occur if and only if $z \in SL_2\mathbb{Z}.i$ or $z \in SL_2\mathbb{Z}\zeta_3$

i.e. $\zeta_3 = e(1/3) = \dfrac{-1+\sqrt{-3}}{2}$.

2/6

Proof ~~Bla~~ Already saw that these groups are finite.

Recall $(SL_2\mathbb{R})_z$ is conjugate to $SO_2(\mathbb{R}) \simeq S^1$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto \cos\theta + i\sin\theta = e^{i\theta}.$$

So any finite subgroup of $(SL_2\mathbb{R})_z$ is cyclic:

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x+iy.$$

[Indeed, given a finite group $G \leq S^1$, let $z_0 = e^{i\theta_0} \in G$ with $\theta_0 \in [0, 2\pi)$ minimal, Then z_0 generates]

Given $z \in \mathcal{H}$, we may assume $z \in \overline{\mathcal{D}}$ ← closure, i.e. $|\mathrm{Re}\, z| \leq 1/2$, $|z| \geq 1$.

Solve $g.z = z$ for $z \iff cz^2 + (d-a)z - b = 0$.

where g generates $(SL_2\mathbb{Z})_z$.

If $g \neq \pm \mathrm{Id}$, and $gz = z$, then g is elliptic, i.e. $\underset{|a+d|}{|\mathrm{tr}(g)|} < 2$.

So two case: $\mathrm{tr}(g) = 0$ or $\mathrm{tr}(g) = \pm 1$

First case: The minimal polynomial for g is $X^2 + 1$ [Cayley-Hamilton + irreducible over $\mathbb{R}$.]

So g satisfies the rules $g^2 = -\mathrm{Id}$, so

$\{\mathrm{Id}, g, -\mathrm{Id}, -g\}$ is a finite cyclic group of order 4.

Replacing g by $-g$, we can assume $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\boxed{c \geq 0.}$.

$\det g = 1 = -a^2 - bc$, so $c > 0$ and:

$$\begin{aligned}
0 &= cz^2 + (d-a)z - b \\
&= (cz)^2 - 2a(cz) - bc \\
&= (cz)^2 - 2a(cz) + 1 + a^2 \\
&= Z^2 - 2aZ + (1+a^2), \text{ so}
\end{aligned}$$

$Z = cz$

$Z = a \pm i$

$$\boxed{z = \frac{a \pm i}{c}}.$$

But $z \in \overline{\mathcal{D}}$, so $\mathrm{Im}\, z \geq \frac{\sqrt{3}}{2}$, so $\frac{1}{c} \geq \frac{\sqrt{3}}{2}$, so $c \leq \frac{2}{\sqrt{3}}$, so $c \leq 1$ i.e. $c = 1$.

$|\mathrm{Re}\, z| \leq 1/2$, so $a = 0$, so $\boxed{z = i}$.

Second Case: $\mathrm{Tr}(g) = \pm 1$.

Swapping g for $-g$ (possibly), we can assume $\mathrm{tr}(g) = 1$.

Thus g has characteristic polynomial

$$X^2 - X + 1.$$

This is irreducible over $\mathbb{R}$, and so by Cayley-Hamilton it is the minl poly of g.

So g satisfies $g^2 - g + Id = 0$, so

$$\{Id, g, g^2 = g - Id, g^3 = -Id, g^4 = -g, g^5 = -g^2\}$$

is a finite cyclic group of order 6 generated by g.

$\det g = 1 = a(1-a) - bc$, so $c \neq 0$, $c > 0$.

$$\begin{aligned} 0 &= cz^2 + (1-2a)z - b \\ &= (cz)^2 + (1-2a)(cz) + 1 - (1-a)a \\ &= Z^2 + (1-2a)Z + (1-a+a^2) \quad \text{with } Z = cz. \end{aligned}$$

So $Z = \dfrac{2a-1 \pm \sqrt{(1-2a)^2 - 4(1-a+a^2)}}{2} = \dfrac{2a-1 \pm \sqrt{-3}}{2}$

so $z = \dfrac{2a-1 \pm \sqrt{-3}}{2c}$.

Again: $\operatorname{Im} z \geq \dfrac{\sqrt{3}}{2} \Rightarrow \dfrac{\sqrt{3}}{2c} \geq \dfrac{\sqrt{3}}{2} \Rightarrow c = 1$

$|\operatorname{Re}(z)| \leq 1/2 \Rightarrow \dfrac{|2a-1|}{2c} \leq 1/2$ so $a = 0, 1$, ie

$z = \dfrac{-1+\sqrt{-3}}{2}$ or $\dfrac{1+\sqrt{-3}}{2}$, ie $z = \zeta_3$ or $-\bar{\zeta}_3$. QED.

COROLLARY
$SL_2\mathbb{Z}_i = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$ has order 4

$SL_2\mathbb{Z}_{\zeta_3} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle$ has order 6.

$SL_2\mathbb{Z}_z = \langle -Id \rangle$ has order 2, $z \notin SL_2\mathbb{Z}\, i$ or $SL_2\mathbb{Z}\, \zeta_3$

COROLLARY
$Y(\Gamma)$ has only finitely many elliptic points.

COROLLARY
For $q \geq 4$, $\Gamma(q)_z = \{Id\}$ $\forall z \in \mathcal{H}$.

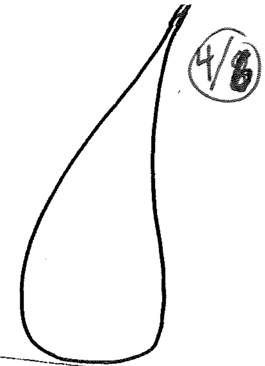
Proof: For $\gamma \in \Gamma(q)$, $tr(\gamma) \equiv 2\ (q)$

$2 \not\equiv 0, \pm 1$ mod q if $q > 3$, so $|tr(\gamma)| \geq 2$.

Defn A Riemann Surface is a connected complex manifold of complex dimension 1.

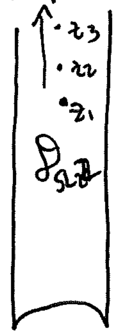
FACT (not proven or used in this course)

$Y(\Gamma)$ is a non-compact Riemann surface.



$Y(SL_2\mathbb{Z}) \cong \mathbb{C}$

Compactification The orbits $(SL_2\mathbb{Z})\cdot z_n$ for points $z_n \in \mathcal{H}$ with $y_n \to \infty$ have no convergent subsequences



Note that $\mathcal{D}_{SL_2\mathbb{Z}}$ is very thin: Let $L(t) = x(t)+iy(t) = (1-t)x_1 + tx_2 + iy$.

$$d_h(x_1+iy, x_2+iy) \le \int_0^1 \frac{\sqrt{(x'(t))^2+(y'(t))^2}}{y(t)}\,dt$$

$$= \int_0^1 \frac{x_2-x_1}{y_1}\,dt = \frac{x_2-x_1}{y} \to 0 \text{ as } y \to \infty$$

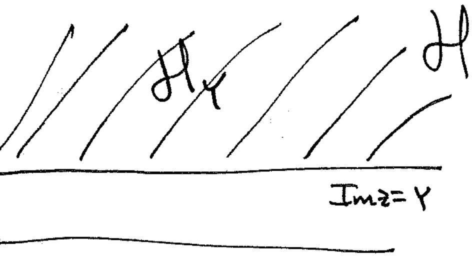
This suggests we use the one-point compactification: $X(1) \stackrel{\text{def}}{=} Y(1) \cup \{\infty\}$.

Topology: It suffices to describe the neighborhoods of ∞

By definition of the 1-point compactification, the neighborhoods of ∞ are complements in $X(1)$ of compact subsets of $Y(1)$.

Basis of neighborhoods of ∞:

Let $\mathcal{H}_Y = \{z \in \mathcal{H} : \operatorname{Im} z > Y\}$, $Y > 0$.



Let $\mathcal{U}_{\infty,Y} = \mathcal{H}_Y \cup \{\infty\}$.

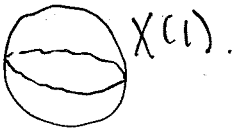
When we were constructing the fundamental domain $\mathcal{D}_{SL_2\mathbb{Z}}$, we showed:

Given $z \in \mathcal{U} \subseteq \mathcal{H}$, $\mathcal{U}$ an open neighborhood with compact closure, there exists $Y > 0$, (depending on $z, \mathcal{U}$) such that

$$\forall \gamma \in SL_2\mathbb{Z}, \quad \gamma\mathcal{H}_Y \cap \mathcal{U} = \emptyset \qquad \text{Disjoint!}$$

Indeed, Recall we showed $\operatorname{Im}(SL_2\mathbb{Z}\cdot z) < \infty$.

Therefore $X(SL_2\mathbb{Z})$ is a compact, connected, Hausdorff topological space.



The map $\mathrm{SL}_2\mathbb{Z}_\infty \backslash \mathcal{H}_Y \longrightarrow \pi_{\mathrm{SL}_2\mathbb{Z}}(\mathcal{H}_Y)$ is a homeomorphism for $Y>1$

From the explicit $\mathcal{D}_{\mathrm{SL}_2\mathbb{Z}}$

$$q_\infty : \mathcal{H} \to \mathbb{D} \qquad \mathbb{D} = \text{open unit disk in } \mathbb{C}.$$

$$z \longmapsto e(z) = e^{2\pi i z}$$

is a homeomorphism locally at ∞ : $q_\infty : \pi_{\mathrm{SL}_2\mathbb{Z}}(U_{\infty,Y}) \simeq D(0, e^{-2\pi Y})$

This is called a "local uniformizer at ∞".

i.e. a homeomorphism from a nbhd of ∞ into $\mathbb{D} \subseteq \mathbb{C}$.

~~This~~ <u>FACT</u> this makes $X(1)$ a <u>compact</u> Riemann ~~sphere~~ surface.

<u>Cusps</u>: More generally, we compactify $Y(\Gamma)$:

Observe that $(\mathrm{SL}_2\mathbb{Z}).\infty = \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{R})$

the <u>Rational</u> projective line.

Indeed, ∞ obviously $\subseteq \mathbb{P}^1(\mathbb{Q})$.

Let $\frac{a}{c} \in \mathbb{Q}$ in lowest terms, so $(a,c)=1$, and by Bézout's formula

$$\exists\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{Z}, \quad \text{and} \quad \gamma \cdot \infty = \frac{a}{c}.$$

Thus, for $\Gamma \leq \mathrm{SL}_2\mathbb{Z}$ of finite index, Γ acts on $\mathbb{P}^1(\mathbb{Q})$.

We decompose into Γ-orbits. These are called <u>$\mathrm{Cusps}(\Gamma)$</u>.

$$\mathbb{P}^1(\mathbb{Q}) = \bigsqcup_{\mathfrak{a} \in \mathrm{Cusps}(\Gamma)} \Gamma . x_{\mathfrak{a}},$$

where $x_{\mathfrak{a}}$ is any choice of representative in $\mathbb{P}^1(\mathbb{Q})$ for the cusp $\mathfrak{a}$.

Let $\hat{\mathcal{H}} \stackrel{\text{def}}{=} \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the "extended upper half plane".

We set $X(1) = \mathrm{SL}_2\mathbb{Z} \backslash \hat{\mathcal{H}} = \mathrm{SL}_2\mathbb{Z} \backslash \mathcal{H} \cup \infty$

and if $\Gamma \leq \mathrm{SL}_2\mathbb{Z}$ is of finite index $X(\Gamma) = \Gamma \backslash \mathcal{H} \sqcup \Gamma \backslash \mathrm{SL}_2\mathbb{Z}.\infty$

$= {}_{\Gamma}\backslash \mathcal{H} \sqcup \mathrm{Cusps}(\Gamma)$.

We define a Hausdorff topology on $\hat{\mathcal{H}}$ by giving a neighborhood basis for each $x \in \mathbb{P}^1(\mathbb{Q})$.

Let $U_{x,Y}$, $Y > 0$ be

$$U_{\infty,Y} = \mathcal{H}_Y \cup \infty \quad \text{if } x = \infty;$$

$$U_{x,Y} = \sigma . U_{\infty,Y} \text{ for any } \sigma \in SL_2\mathbb{Z} \text{ such that } \sigma.\infty = x.$$

Such a σ is called a "scaling matrix for $\mathfrak{a} = \Gamma.x$".
usually denoted $\sigma_{\mathfrak{a}}$, but is a choice.

The $U_{x,Y}$ are ~~[illegible]~~ hyperbolic disks in $\mathcal{H}$ tangent to $\mathbb{R}$ at x.

Note it is NO longer true that the stabilizers $(SL_2\mathbb{Z})_z$ of $z \in \hat{\mathcal{H}}$ are finite: We have $(SL_2\mathbb{Z})_\infty = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$, and if $x \in \mathbb{P}^1(\mathbb{Q})$ then $(SL_2\mathbb{Z})_x$ is conjugate to this. However, there is an analogue of Thms 1, 2.

Prop. Let Γ be a congruence subgroup of $SL_2\mathbb{Z}$, $x, y \in \mathbb{P}^1(\mathbb{Q})$ and $z \in \mathcal{H}$.

1. For any $z \in U$ nbhd with compact closure, $Y > 0$ the set $\{\gamma \in \Gamma, \gamma U_{x,y} \cap U \neq \emptyset\}$ is finite for U and $\frac{1}{Y}$ sufficiently small. In fact, it is empty if U, $1/Y$ are sufficiently small.
2. If $y \notin \Gamma x$ then
$$\{\gamma \in \Gamma : \gamma U_{x,Y} \cap U_{y,Y} \neq \emptyset\} \text{ is finite for any } Y > 0$$
and empty if $Y > 1$.
3. If $Y > 1$,
$$\{\gamma \in \Gamma : \gamma U_{x,Y} \cap U_{x,Y} \neq \emptyset\} = \Gamma_x.$$

Proof: Exercise!