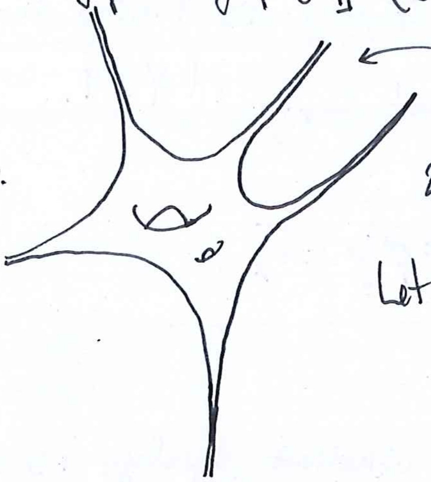
Classical Modular Forms Lecture 11 2.11.2017

Recall $\hat{\mathcal{H}} = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, $\Gamma \leq SL_2\mathbb{Z}$ finite index

$Y(\Gamma)$:



$\mathrm{Cusps}(\Gamma) = \Gamma$ - orbits on $\mathbb{P}^1(\mathbb{Q})$

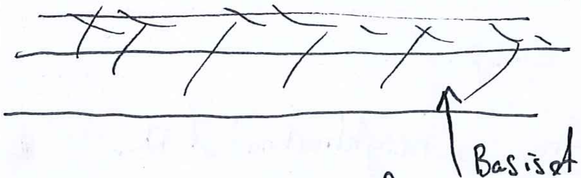
Compactification:

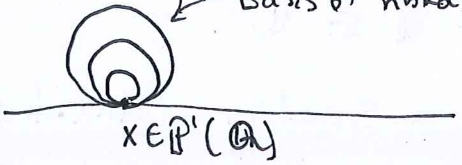
Let $X(\Gamma) = \Gamma \backslash \hat{\mathcal{H}} = \Gamma \backslash \mathcal{H} \cup \Gamma \backslash SL_2\mathbb{Z}.\infty$

$= Y(\Gamma) \sqcup \mathrm{Cusps}(\Gamma)$.

To define a Hausdorff topology on $X(\Gamma)$, we first need such a topology on $\hat{\mathcal{H}}$.

For each $x \in \mathbb{P}^1(\mathbb{Q})$, we need to define a neighborhood basis:





Let

$$U_{x,Y} = \begin{cases} \text{if } x=\infty & \mathcal{H}_Y \cup \{\infty\} \\ \text{if } x \neq \infty & \sigma U_{\infty,Y} \end{cases}$$

where σ is any matrix $\in SL_2\mathbb{Z}$ such that $\sigma.\infty = x$.

Given a cusp $\mathfrak{a}$, ~~on~~ $SL_2\mathbb{Z}$ acts transitively on Cusps Γ, and a $\sigma \in SL_2\mathbb{Z}$ such that $\sigma\infty = \mathfrak{a}$ is called a 'Scaling matrix' & usually denoted $\sigma_{\mathfrak{a}}$.

Warning: $\sigma_{\mathfrak{a}}$ is far from uniquely defined.

Note: Unlike $z \in \mathcal{H}$, the stabilizers of $x \in \mathbb{P}^1(\mathbb{Q}) \subseteq \hat{\mathcal{H}}$ are not finite.

We have $(SL_2\mathbb{Z})_\infty = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$, and $(SL_2\mathbb{Z})_x$, $x \in \mathbb{P}^1(\mathbb{Q})$ is conjugate to $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$.

However, we do have the following analogue of theorem 2 from Tuesday's course:

PROPOSITION Let Γ be a congruence subgroup of $SL_2\mathbb{Z}$, $x,y \in \mathbb{P}^1(\mathbb{Q})$ and $z \in \mathcal{H}$.

① For any $z \in U$, a nbhd with compact closure, $Y > 0$, the set

$$\{\gamma \in \Gamma, \gamma U_{x,Y} \cap U \neq \emptyset\}$$ is finite for U and $1/Y$

sufficiently small, and in fact is empty for $U, 1/Y$ suff. small.

② If $y \notin \Gamma x$, then

$$\{\gamma \in \Gamma : \gamma U_{x,Y} \cap U_{y,Y} \neq \emptyset\} \text{ is finite for any } Y>0$$

and empty if $Y>1$.

③ If $Y>1$

$$\{\gamma \in \Gamma : \gamma U_{x,Y} \cap U_{x,Y} \neq \emptyset\} = \Gamma_x$$

PROOF: Exercise!

COROLLARY $X(\Gamma)$ equipped with the Quotient topology is a connected, compact, Hausdorff topological space.
(Another exercise)

Local Uniformizers: Let $\mathfrak{a} = \Gamma . x$ be a cusp.

A "local uniformizer" for $\mathfrak{a}$ is a homeomorphism from a neighborhood of $\mathfrak{a}$ to the open unit disk $\mathbb{D} \subseteq \mathbb{C}$.

We can define them as follows: Let $\sigma_{\mathfrak{a}}$ be a scaling matrix for $\mathfrak{a}$ i.e. a matrix in $SL_2 \mathbb{Z}$ s.t. $\sigma_{\mathfrak{a}} \infty = \mathfrak{a}$.

By the previous proposition, for $Y>1$, $x \in \mathfrak{a}$

$$U_{\mathfrak{a},Y} \overset{\text{def}}{=} \pi_\Gamma(U_{x,Y}) \simeq \Gamma_{\mathfrak{a}} \backslash U_{x,Y} \simeq \sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} \backslash U_{\infty,Y}$$

If Γ is a congruence subgroup, then

$$\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} \subset (SL_2 \mathbb{Z})_\infty = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}} \text{ is finite index.}$$

Better: $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}$ is itself a congruence subgroup.

If $\Gamma(q) \subseteq \Gamma$, then $\sigma_{\mathfrak{a}} \Gamma(q) \sigma_{\mathfrak{a}}^{-1} = \Gamma(q)$

so $\Gamma(q) \subseteq \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}$

And $\Gamma_\infty(q)$ is index $\leq 2q$ in $(SL_2 \mathbb{Z})_\infty$, so $\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}}$ finite index in $(SL_2 \mathbb{Z})_\infty$.

Thus $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{cases} \pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^{\mathbb{Z}} \\ \text{or} \\ \begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^{\mathbb{Z}}\end{cases}$ for a unique $h \in \mathbb{Z}_{>0}$.

Defn The integer h only depends on the cusp $\mathfrak{a}$ and is called the width of $\mathfrak{a}$.

Then, let $q_{\mathfrak{a}} : \hat{\mathcal{H}} \longrightarrow \mathbb{D}$ be defined by

$$z \longmapsto q_\infty\left(\frac{\sigma_{\mathfrak{a}}^{-1}z}{h}\right) = e\left(\frac{\sigma_{\mathfrak{a}}^{-1}z}{h}\right)$$

~~This only depends~~ restricts to a homeomorphism:

$$q_{\mathfrak{a}} : U_{\mathfrak{a},Y} = \Gamma_{\mathfrak{a}}\backslash U_{x,Y} \xrightarrow{\sim} D(0, e^{\frac{-2\pi Y}{h}})$$

indeed, let $\gamma \in \Gamma_{\mathfrak{a}}$, then $\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} = \pm\begin{pmatrix}1 & hk\\ 0 & 1\end{pmatrix}$ for some $k \in \mathbb{Z}$.

So $q_{\mathfrak{a}}(\gamma z) = e\left(\frac{\pm\begin{pmatrix}1 & hk\\ 0 & 1\end{pmatrix}\sigma_{\mathfrak{a}}^{-1}z}{h}\right) = e\left(\frac{\sigma_{\mathfrak{a}}^{-1}z + hk}{h}\right) = q_{\mathfrak{a}}(z)$.

Exercise Show that a set of representatives for $\mathrm{Cusp}(\Gamma_0(q))$ is given by the fractions $\left\{\frac{u}{v} : v \mid q,\ 0 < u \leq (v, q/v)\right\}$.

compute their widths.

[Using these local uniformizers at cusps, $X(\Gamma)$ is a compact Riemann surface:  . If $\Gamma' \leq \Gamma$, we have projections $X(\Gamma') \to X(\Gamma)$ where the map is a map of Riemann surfaces]

Hyperbolic measure on $Y(\Gamma)$:

The measure $d\mu(z) = \frac{dx\,dy}{y^2}$ on $\mathcal{H}$ is $SL_2\mathbb{Z}$-invariant, and so descends to $Y(\Gamma)$. We denote it by $d\mu_\Gamma(\Gamma.z) = \frac{dx\,dy}{y^2}$.

Let f be a μ_Γ-integrable function on $Y(\Gamma)$. We write

$$\mu_\Gamma(f) = \int_{Y(\Gamma)} f(\Gamma.z)\,d\mu_\Gamma(\Gamma.z) = \int_{Y(\Gamma)} f(\Gamma.z)\frac{dx\,dy}{y^2}.$$

More concretely: The functions $Y(\Gamma)$ are canonnically identified with the functions on $\mathcal{H}$ which are Γ-invariant via:

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \tilde{f} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ \{Y(\Gamma) \to \mathbb{C}\} & \{\mathcal{H} \to \mathbb{C}\} \end{tikzcd} \qquad \text{via } \tilde{f}(z) := f(\Gamma . z).$$

and a locally μ_Γ - integrable function on $Y(\Gamma)$ is a locally $\mu_{\mathcal{H}}$-integrable Γ-invariant function on $\mathcal{H}$.

We have for such functions : $\mu_\Gamma(f) = \int_{\mathcal{D}_\Gamma} \tilde{f}(z)\, d\mu(z)$.

Continuous bounded functions are integrable. Indeed:

$$\mu_\Gamma(1) = \int_{\mathcal{D}_\Gamma} 1\, d\mu(z) = \sum_{\gamma_i \in \frac{SL_2\mathbb{Z}}{\Gamma}} \int_{\gamma_i \mathcal{D}_{SL_2\mathbb{Z}}} 1\, d\mu(z) = [SL_2\mathbb{Z} : \Gamma] \int_{\mathcal{D}_{SL_2\mathbb{Z}}} 1\, d\mu(z).$$

And $$\int_{\mathcal{D}_{SL_2\mathbb{Z}}} 1 . d\mu(z) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} 1 \frac{dx\,dy}{y^2} = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \arcsin x \Big|_{-1/2}^{1/2} = \pi/6 - (-\pi/6) = \pi/3 < \infty.$$

So $\mathrm{Vol}(Y(\Gamma)) = [SL_2\mathbb{Z} : \Gamma]\, \mathrm{Vol}\, Y(1) = \frac{\pi}{3} [SL_2\mathbb{Z} : \Gamma]$.

<u>Normalization</u>: If $\Gamma' \leq \Gamma$ then $Y(\Gamma') \twoheadrightarrow Y(\Gamma)$

and $L^1(Y(\Gamma)) \hookrightarrow L^1(Y(\Gamma'))$ is an injection.

Said more simply: If f is Γ-invariant, then it is also Γ'-invariant.

For any f, $\mu_{\Gamma'}(f) = [\Gamma ; \Gamma'] \mu_\Gamma(f)$.

So we can define a <u>normalized</u> measure μ_n on space of bounded functions on $\mathcal{H}$ which are invariant by <u>some</u> congruence subgroup Γ by:

$$\mu_n(f) = [SL_2\mathbb{Z} : \Gamma]^{-1} \mu_\Gamma(f).$$

This does <u>not</u> depend on the choice of subgroup: If f is Γ-invt and Γ'-invt then f is $\Gamma'' = \Gamma \cap \Gamma'$ invariant, and

Then $[SL_2\mathbb{Z} : \Gamma''] = [SL_2\mathbb{Z} : \Gamma][\Gamma : \Gamma''] = [SL_2\mathbb{Z} : \Gamma'] \cdot [\Gamma', \Gamma'']$.

So $[SL_2\mathbb{Z} : \Gamma]^{-1} \mu_\Gamma(f) = [SL_2\mathbb{Z} : \Gamma'']^{-1} \mu_{\Gamma''}(f) = [SL_2\mathbb{Z} : \Gamma']^{-1} \mu_{\Gamma'}(f)$.

Back to modular forms. Recall $S_k(\Gamma)$ ~~modular~~ cusp forms for Γ weight k, trivial nebentype character.

Petersson Inner Product:

Γ a congruence subgroup.

The space $S_k(\Gamma)$ of cusp forms of weight k for Γ has a natural inner product:

For $f_1, f_2 \in S_k(\Gamma)$ the function $y^k f_1(z) \overline{f_2(z)}$ is Γ-invariant, and bounded. We define the normalized Petersson inner product.

$$\langle f_1, f_2 \rangle = \int_{Y(\Gamma)} f_1(z) \overline{f_2(z)} \, y^k \, \cancel{dxdy} \, \mu_n(z).$$

Does not depend on the choice of Γ.

We also write $\| f \|_{L^2} = \langle f, f \rangle^{1/2}$ the L^2-norm.

↯ Warning: Both $\mu_n(z)$ and $\mu_\Gamma(z)$ are used in the litterature to define inner products, so careful.

Finite-dimensionality of space of modular forms:

Recall: The subspace $S_k(\Gamma)$ has codimension at most $[SL_2\mathbb{Z} : \Gamma]$ in $M_k(\Gamma)$. So it suffices to prove $\dim S_k(\Gamma) < \infty$.

Main input:

Theorem (Sobolev estimate)

For any $f \in S_k(\Gamma)$

$$\| f \|_\infty \overset{\text{def}}{=} \sup_{z \in \mathcal{H}} F(z) = \sup_{z \in \mathcal{H}} y^{k/2} |f(z)| \leq C(\Gamma, k) \| f \|_{L^2}$$

Vaguely, this says the mass of f cannot escape towards the cusp as we run over $f \in S_k(\Gamma)$.

Proof of finiteness assuming the Theorem:

Let $\{f_1, \dots\} \subseteq S_k(\Gamma)$ be an orthonormal basis with respect to the normalized Petersson inner product.

Then for any $z \in \mathcal{H}$, $\{\alpha_i\} \in \mathbb{C}$

$$y^{k/2} \left| \sum_i \alpha_i f_i(z) \right| \underset{\substack{\uparrow \\ \text{Sobolev}}}{\le} C(\Gamma, k) \| \alpha_i f_i \|_{L^2} \underset{\substack{\uparrow \\ \text{Pythagorean} \\ \text{Theorem}}}{=} C(\Gamma, k) \left(\sum_i |\alpha_i|^2 \right)^{1/2}$$

Let $\alpha_i = y^{k/2} \overline{f_i(z)}$

Then for any $z \in \mathcal{H}$

$$y^k \sum_i |f_i(z)|^2 \le C(\Gamma, k) \left(y^k \sum_i |f_i(z)|^2 \right)^{1/2}$$

$$\Rightarrow y^k \sum_i |f_i(z)|^2 \le C(\Gamma, k)^2 \quad \underline{\text{for any}}\ z \in \mathcal{H}.$$

We have $\langle f_i, f_i \rangle = 1$, so $\int_{\mathcal{F}_\Gamma} |f_i(z)|^2 y^k \frac{dx\,dy}{y^2} = [SL_2\mathbb{Z} : \Gamma]$

So integrate $y^k \sum_i |f_i(z)|^2$ over $\mathcal{F}_\Gamma$:

$$\sum_i [SL_2\mathbb{Z} : \Gamma] \le C(\Gamma, k)^2 \operatorname{Vol} Y(\Gamma)$$

so $$\dim S_k(\Gamma) \le C(\Gamma, k)^2 \frac{\operatorname{Vol} Y(\Gamma)}{[SL_2\mathbb{Z} : \Gamma]} = \frac{\pi}{3} C(\Gamma, k)^2$$

QED

Now we prove the Sobolev estimate.

The main input / idea is in the following lemma:

~~Let~~ Lemma Let $\alpha > 1$ and let f be a function holomorphic on $D(0, \alpha R)$.

For $z \in D(0, R/2)$

$$|f(z)|^2 \ll_\alpha \frac{1}{R^2} \int_{D(0,R)} |f(w)|^2 \, dw$$

PROOF Replacing $f(z)$ by $f(Rz)$, we can assume $R=1$.

Write $f(z) = \sum_{n \geqslant 0} a_n z^n$, which is uniformly and absolutely convergent in ~~$\mathbb{R}$~~ ~~$\mathbb{D}_2$~~ ~~$D(0,R)$~~ $\mathbb{D}$.

We write $w = re(\theta)$

$$\int_{D(0,1)} \Big|\sum_{n \geqslant 0} a_n r^n e(n\theta)\Big|^2 r\,dr\,d\theta = 2\pi \int_0^1 \sum_{n \geqslant 0} |a_n|^2 r^{2n}\, r\,dr$$

$$= 2\pi \sum_{n \geqslant 0} |a_n|^2 \int_0^1 r^{2n+1} dr = 2\pi \sum_{n \geqslant 0} \frac{|a_n|^2}{2n+2}.$$

On the other hand, since $|z| \leq 1/\alpha$, and by Cauchy-Schwarz:

$$|f(z)|^2 \leq \left(\sum_{n \geqslant 0} \frac{|a_n|^2}{2n+2}\right)\left(\sum_{n \geqslant 0} (2n+2)|z|^{2n}\right) = \frac{1}{2\pi}\int_{D(0,1)} |f(w)|^2 dw \cdot \left(\sum_{n \geqslant 0} \frac{(2n+2)}{\alpha^{2n}}\right).$$

PROOF of Sobolev estimate:

Fix a fundamental domain

$$\mathcal{F}_\Gamma = \bigsqcup_{\gamma_i \in \Gamma \backslash SL_2\mathbb{Z}} \gamma_i \mathcal{F}_{SL_2\mathbb{Z}}$$

Cover $\mathcal{F}_\Gamma$ by a finite number of open sets U_j, and we check the inequality on each of these sets separately.

Harder part is to treat the U_j associated to the cusps:

For each cusp $\mathfrak{a}$, let $\sigma_\mathfrak{a} \infty = \mathfrak{a}$ be a scaling matrix $\in SL_2\mathbb{Z}$

let $U_{\mathfrak{a},Y} = \pi_\Gamma(\sigma_\mathfrak{a} \mathcal{H}_Y)$ (as before).

Recall for $Y \geqslant 1$, $\pi_\Gamma(\sigma_\mathfrak{a} \mathcal{H}_Y) \simeq \Gamma_\mathfrak{a} \backslash U_{\infty,Y}$

is parameterized by $q_\mathfrak{a} : U_{\mathfrak{a},Y} \xrightarrow{\sim} D(0, e^{-2\pi Y/h})$

$$z \longmapsto e\left(\frac{\sigma_\mathfrak{a}^{-1} z}{h}\right).$$

We change variables: $Z = X + iY = \sigma_\mathfrak{a}^{-1} z$

So by $SL_2\mathbb{Z}$-invariance of measure:

$$\int_{U_{\mathfrak{a},Y}} y^k |f(z)|^2 \frac{dx\,dy}{y^2} = \int_{\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^{\mathbb{Z}} \backslash \mathcal{H}_Y} Y^k |f|\sigma_{\mathfrak{a}}(Z)|^2 \frac{dX\,dY}{Y^2}$$

Where $f|_{\sigma_{\mathfrak{a}}}(Z) = \sum_{n\geq 1} \lambda_{f|\sigma_{\mathfrak{a}}}(n)\, q_{\mathfrak{a}}(z)^n$

$q_{\mathfrak{a}} = r\,e(\theta)$, and so $r = \exp(-2\pi Y/h)$, so $Y = \frac{h}{2\pi}\log(1/r)$.

$$\text{So} = \left(\frac{h}{2\pi}\right)^{k-2} \int_{D(0,\, e^{-2\pi Y/h})} \Big|\sum_{n\geq 1} \lambda_{f|\sigma_{\mathfrak{a}}}(n)\, r^n e(n\theta)\Big|^2 (\log 1/r)^{k-2} \frac{dr\,d\theta}{r}$$

opening the square:

$$= c(h,k) \sum_{n\geq 1} |\lambda_{f|\sigma_{\mathfrak{a}}}(n)|^2 \int_0^R r^{2n-1} (\log 1/r)^{k-2}\, dr = C_R(h,k) \sum_{n\geq 1} |\lambda_{f|\sigma_{\mathfrak{a}}}(n)|^2$$

.... etc. Finish the proof next week!