Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space $/\mathbb{C}$.

i.e.
- $f \mapsto \langle f, g\rangle$ is linear in f for every fixed $g \in V$
- $\langle f, g\rangle = \overline{\langle g, f\rangle}$
- $\langle f, f\rangle \geq 0$ for all $f \in V$.

We write $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$.

~~Lemma~~ Note that if $\langle f, g\rangle = 0$ then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$

Indeed, $\|f+g\|^2 = \langle f+g, f+g\rangle = \|f\|^2 + \langle f,g\rangle + \langle g,f\rangle + \|g\|^2$

Since $\langle f,g\rangle = 0 \Rightarrow \langle g,f\rangle = 0$

we have $\|f+g\|^2 = \|f\|^2 + \|g\|^2$.

We ~~was~~ say a countable ~~[illegible]~~ set $\{e_1, e_2, \ldots\}$ of V is orthonormal if
$$\langle e_k, e_l\rangle = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

<u>Lemma</u>: If $\{e_k\}_{k=1}^{\infty}$ is orthonormal, and $f = \sum_k a_k e_k$ is a <u>finite</u> sum, then
$$\|f\|^2 = \sum_k |a_k|^2$$

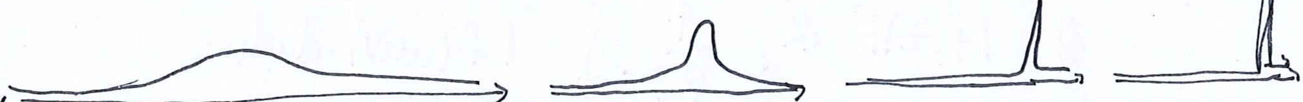
<u>Proof</u>: This follows from the above discussion of the Pythagorean Theorem.

Main input to prove the finite-dimensionality of $S_k(\Gamma)$ is the following:

<u>Theorem</u> (Sobolev estimate)

For any $f \in S_k(\Gamma)$
$$\|f\|_\infty \overset{\text{def}}{=} \sup_{z\in\mathcal{H}} F(z) = \sup_{z\in\mathcal{H}} y^{k/2}|f(z)| \leq C(\Gamma,k)\|f\|_{L^2}$$

Vaguely:



asp

Normalize fcns so that $\|f\|_{L^2} = 1$

The above sequence of functions isn't allowed.

PROOF (of finite-dimensionality).

Suppose $\{f_1, \ldots, f_r\}$ is a finite orthonormal set in $S_k(\Gamma)$ wrt normalized Petersson inner product.

Then for any $z \in \mathcal{H}$, and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$

$$y^{k/2}\left|\sum_{i=1}^{r} \alpha_i f_i(z)\right| \underset{\text{Sobolev}}{\leq} C(\Gamma,k)\left\|\sum_{i=1}^{r}\alpha_i f_i\right\|_{L^2} \underset{\text{Lemma}}{=} C(\Gamma,k)\left(\sum_{i=1}^{r}|\alpha_i|^2\right)^{1/2}$$

But then we can take $\alpha_i = y^{k/2} f_i(z)$ so

$$y^k \sum_{i=1}^{r} |f_i(z)|^2 \leq C(\Gamma,k)\left(y^k\sum_{i=1}^{r}|f_i(z)|^2\right)^{1/2}$$

$$\Rightarrow \quad y^k \sum_{i=1}^{r} |f_i(z)| \leq C(\Gamma,k)^2$$

Now take $\frac{1}{[SL_2\mathbb{Z}:\Gamma]}\int_{\mathcal{D}_\Gamma} \cdot \, \frac{dx\,dy}{y^2}$ of both sides

LHS $= r$ since f_i are orthonormal for the Petersson inner prod.

RHS $= \frac{\pi}{3}$ ~~since~~ $C(\Gamma,k)^2$, since we showed $\frac{1}{[SL_2\mathbb{Z}:\Gamma]}\int_{\mathcal{D}_\Gamma}\frac{dx\,dy}{y^2}$ does not depend on Γ, and for $\Gamma = SL_2\mathbb{Z}$ equals $\pi/3$.

Therefore the size of an orthonormal set is bounded by $\leq C(\Gamma,k)^2\frac{\pi}{3}$.

So the size of any basis is also bounded (Gram-Schmidt process)

Now we prove the Sobolev estimate.

Basic idea is contained in the following lemma:

Lemma: Let $\alpha > 1$, and let f be a holomorphic function on $D(0,\alpha R)$.

For $z \in D(0, R/\alpha)$

$$|f(z)|^2 \ll_\alpha \frac{1}{R^2}\int_{D(0,R)} |f(u+iv)|\,du\,dv$$

$D(0,R)$ ← complex ~~open~~ disk of radius R

<u>Proof</u> Replacing $f(z)$ by $f(Rz)$, we can assume $R=1$.

f is holomorphic in $D(0,\alpha)$, so we can write a Taylor series:

$$f(z) = \sum_{n\geq 0} a_n z^n$$

and this series converges uniformly in the disk.

Let us write $w = re(\theta) = u+iv$

Then

$$\int_{D(0,1)} |f(w)|^2 \, du\,dv = \int_{D(0,1)} \Big|\sum_{n\geq 0} a_n r^n e(n\theta)\Big|^2 r\,dr\,d\theta$$

$$= 2\pi \int_0^1 \sum_{n\geq 0} |a_n|^2 r^{2n+1} dr = 2\pi \sum_{n\geq 0} \frac{|a_n|^2}{2n+2}$$

Since $|z| \leq 1/\alpha$ we have by the Cauchy - Schwarz inequality:

$$|f(z)|^2 \leq \left(\sum_{n\geq 0} \frac{|a_n|^2}{2n+2}\right)\left(\sum_{n\geq 0} (2n+2)|z|^{2n}\right) = \frac{1}{2\pi}\int_{D(0,1)} |f(w)|^2 dw \left(\sum_{n\geq 0} \frac{2n+2}{\alpha^{2n}}\right)$$

converges.

$$\leq \frac{\sum_{n\geq 0} \frac{2n+2}{\alpha^{2n}}}{2\pi} \int_{D(0,1)} |f(w)|^2 dw.$$

<u>Proof of Sobolev estimate:</u>

Strategy is to cover $X(\Gamma)$ by a finite number of open sets. We use the lemma away from the cusps, and use the local uniformizers $q_{\mathfrak{a}}$ we previously introduced at the cusps.

First, consider an open neighborhood at a cusp $\mathfrak{a}$.

Let $\sigma_{\mathfrak{a}} \in SL_2\mathbb{Z}$ be a scaling matrix for $\mathfrak{a}$, ie $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$

Let $U_{\mathfrak{a},Y} = \pi_\Gamma(\sigma_{\mathfrak{a}} \mathcal{H}_Y)$ be a standard open neighborhood of $\mathfrak{a}$ in $X(\Gamma)$.

Recall from HW that if $Y>1$ $\quad U_{\mathfrak{a},Y} \underset{\text{homeomorphism}}{\cong} {}^{\sigma}\!\mathcal{U}_{\infty,Y} / \Gamma_{\mathfrak{a}}$

and so $q_{\mathfrak{a}} : U_{\mathfrak{a}} \xrightarrow{\sim} D(0, e^{-2\pi Y/h})$

$$z \longmapsto e\left(\frac{\sigma_{\mathfrak{a}}^{-1} z}{h}\right)$$

Consider $\displaystyle\int_{U_{\mathfrak{a},Y}} y^k |f(z)|^2 \frac{dx\,dy}{y^2}$. Change variables $Z = X + iY = \sigma_{\mathfrak{a}}^{-1} z$

$$= \int_{\begin{pmatrix}1 & h\\ & 1\end{pmatrix}^{\mathbb{Z}} \backslash \mathcal{H}_Y} Y^k |f|\sigma_{\mathfrak{a}}(Z)|^2 \frac{dX\,dY}{Y^2} \quad \text{by } SL_2\mathbb{Z} \text{ invariance of measure.}$$

We can expand $f|_{\sigma_{\mathfrak{a}}}(Z) = \sum_{n \geq 1} \lambda_{f|\sigma_{\mathfrak{a}}}(n)\, q_{\mathfrak{a}}(z)^n$ in Fourier series

"The Fourier expansion" at the cusp $\mathfrak{a}$

Let $q_{\mathfrak{a}} = r e(\theta)$, so $r = \exp\left(-\frac{2\pi Y}{h}\right)$, so $Y = \left(\frac{h}{2\pi}\right) \log 1/r$

$$\text{so} \int_{U_{\mathfrak{a},Y}} y^k |f(z)|^2 \frac{dx\,dy}{y^2} = \left(\frac{h}{2\pi}\right)^{k-2} \int_{D(0, e^{-2\pi Y/h})} \Big|\sum_{n\geq 1} \lambda_{f|\sigma_{\mathfrak{a}}}(n) r^n e(n\theta)\Big|^2 (\log 1/r)^{k-2}\, r\,dr\,d\theta.$$

$$= \left(\frac{h}{2\pi}\right)^{k-2} \sum_{n \geq 1} |\lambda_{f|\sigma_{\mathfrak{a}}}(n)|^2 \underbrace{\int_0^{e^{-2\pi Y/h}} r^{2n+1} (\log 1/r)^{k-2}\, dr}_{C_{Y,k,h}(n)}$$

Now we proceed as we did in the lemma.

Let $z \in U_{\mathfrak{a},2Y} \subseteq U_{\mathfrak{a},Y}$.

Since $|q_{\mathfrak{a}}| = r \leq e^{-\frac{4\pi Y}{h}}$

$$y^k |f(z)|^2 = Y^k |f|_{\sigma_{\mathfrak{a}}}(Z)|^2 = Y^k \Big|\sum \lambda_{f|\sigma_{\mathfrak{a}}}(n) r^n e(n\theta)\Big|^2$$

$$\leq \left(\frac{h}{2\pi}\right)^k (\log 1/r)^k \left(\sum_{n\geq 1} |\lambda_{f|\sigma_{\mathfrak{a}}}(n)|^2 C_{Y,k,h}(n)\right)\left(\sum_{n\geq 1} \frac{r^{2n}}{C_{Y,k,h}(n)}\right)$$

We have $C_{Y,k,h}(n) \geq \dfrac{1}{2n} e^{-\frac{4\pi Y n}{h}}$

so that $\sum_{n\geq 1} \frac{r^{2n}}{c_{\gamma,k,h}(n)}$ converges, and is bounded by $\ll \frac{r}{R}$

So for $z \in U_{\alpha,Y}$

$$y^k |f(z)| \ll_{h,k} \int_{U_{\alpha,Y}} y^k |f(z)|^2 \frac{dx\,dy}{y^2} (\log 1/r)^2 \frac{r}{R} \ll_{h,k} \|f\|_{L^2}^2$$

Now consider $Y(\Gamma)_{<2Y}$, the complement of $\bigcup_{\alpha \in \text{cusps}} U_{\alpha,2Y}$.

So there exists $K_Y \subseteq \mathcal{H}$ a <u>compact</u> subset such that

$\pi_\Gamma(K_Y)$ contains $Y(\Gamma)_{\leq 2Y}$.

Then we can cover K_Y by a finite number of disks.

That is, $\exists\, z_1, \dots, z_m \in \mathcal{H}$, $R>0$, $\alpha>1$ such that

$$K_Y \subseteq \bigcup_{i=1}^m D(z_i, R/\alpha) \subseteq \bigcup_{i=1}^m D(z_i, \alpha R) \subseteq \mathcal{H}.$$

By the lemma, for each z_i, $z \in D(z_i, R/\alpha)$

$$y^k |f(z)|^2 \ll |f(z)|^2 \ll \int_{D(z_i,R)} |f(w)|^2\, du\, dv \ll \int_{D(z_i,R)} v^k |f(w)|\frac{du\,dv}{v^2}$$

constants depend on R, α, z_i

Therefore, $y^k |f(z)|^2 \ll \|f\|_{L^2}^2$. QED.

In general: There exists a constant c such that

$\psi(q) = [SL_2\mathbb{Z} : \Gamma_0(q)]$

$\dim M_k(\Gamma) \leq$ ~~$c(k+1)(\mathrm{vol}\,\mathcal{D}_\Gamma+1)$~~ $c(k+1)(\mathrm{vol}\,\mathcal{D}_\Gamma+1)$ $\forall k, \Gamma$.

$\dim M_k(q,\chi) = \frac{(k-1)\psi(q)}{12} + O(\sqrt{kq})$

By studying the Riemann surface structure more carefully, using the Riemann-Roch theorem, one can prove <u>exact</u> formulas for $\dim M_k(\Gamma)$ in terms of the genus of $X(\Gamma)$ and the orders of stabilizers of elliptic points of Γ.

Next Section: Hecke Theory & L-fcns.

We already saw that finite-dimensionality $\Rightarrow$ Many interesting arithmetic identities.

The next section is motivated by a 1916 paper of Ramanujan where he made two remarkable and deep conjectures about the Δ-fcn:

Recall
$$\Delta(z) = \frac{E_4^3 - E_6^2}{1728} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n$$
where ~~exist~~ $q = e(z)$

You proved these in exercises.

$\Delta \in S_{12}(SL_2\mathbb{Z})$, and $\dim S_{12}(SL_2\mathbb{Z}) = 1$.

Ramanujan conjectured ① $\tau(mn) = \tau(m)\tau(n)$ if $(m,n)=1$.

② $|\tau(n)| \leq d(n)\, n^{11/2}$,

where $d(n) = \sum_{d \mid n} 1$.

① was proved as a consequence of Hecke's theory in the 1930s

② Note we showed $\tau(n) \ll n^6$, since Δ is cuspidal.

Deligne proved ② in 1974 using algebro-geometric methods (Weil conjectures) it is one of the crowning achievements of mathematics and goes way beyond the content of this course.

Here we develop the Hecke theory to prove ①.

Hecke Theory only pertains to integer weight k so we suppose this until further notice.

Fix k, q a weight and level, χ a Dirichlet character mod q satisfying $\chi(-1) = (-1)^k$.

Define
$$T(n) f(z) = \frac{1}{n} \sum_{ad=n} \chi(a)\, a^k \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right)$$
for $n \geq 1$ integer.

(Warning: $T(n)$ only depends on χ, so implicitly on q)

PROPOSITION For any $n \geq 1$, $T(n)$ acts on modular forms and cusp forms. That is to say, the following are linear maps:

7/8

$$T(n) : M_k(q,\chi) \longrightarrow M_k(q,\chi)$$
$$T(n) : S_k(q,\chi) \longrightarrow S_k(q,\chi)$$

To prove this, we need to re-interpret $T(n)$ as an action of some group acting via the slash operator.

Let $\Gamma = SL_2\mathbb{Z}$ for the time being.

Consider the set

$$M_n = \{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \gamma \in M_2(\mathbb{Z}), \det \gamma = n \}.$$

So eg. $M_1 = \Gamma$.

Γ acts on M_n on the left and on the right by multiplication

Lemma The set

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = n, \ 0 \leq b < d \right\}$$

forms a complete set of right coset representatives of M_n mod Γ

that is,
$$M_n = \bigcup_{\rho \in \Delta_n} \Gamma \rho.$$

PROOF Let $\cancel{\rho}\ r = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in M_n$

There exist γ, δ such that $\gamma a + \delta c = 0$ and $(\gamma, \delta) = 1$

Indeed, take $\gamma = \dfrac{c}{(a,c)}$, $\delta = \dfrac{-a}{(a,c)}$.

Hence, $\exists\ \tau = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in SL_2\mathbb{Z}$ and $\tau r = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

with $da = n$, $d > 0$

multiplying on the left by $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, we can bring b to $0 \le b < d$.

8/8

So, the union $\bigcup_{u \in \mathbb{Z}}$ cosets covers M_n.

For $\rho_1, \rho_2 \in \Delta_n$, the cosets $\Gamma\rho_1, \Gamma\rho_2$ are either disjoint or $\rho_1 = \rho_2$.

Indeed, suppose they are not disjoint. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} \alpha a & \alpha b + \beta d \\ \gamma a & \gamma b + \delta d \end{pmatrix}$$ but $\gamma a = 0$, so $\gamma = 0$

$$\parallel$$

$$\begin{pmatrix} \alpha a & \alpha b + \beta d \\ 0 & \delta d \end{pmatrix}$$ but $\alpha a \delta d = n$, $ad = n$

so $\alpha\delta = 1$, since $d > 0$

$\alpha = 1$, $\delta = 1$

Since $0 \le b + \beta d < d$

we must have $\beta = 0$

Then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and so $\rho_1 = \rho_2$.

QED.