Let $\Delta(z) = \dfrac{E_4^3 - E_6^2}{1728} = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n$

where $q = e(z)$.

Δ spans the 1-diml space $S_{12}(SL_2\mathbb{Z})$.

Ramanujan: $\tau(mn) = \tau(m)\tau(n)$ $(m,n)=1$.

We prove this as a consequence of Hecke Theory:

Suppose k is an integer until further notice.

Let k a weight, q a level, χ a Dirichlet character mod q satisfying $\chi(-1) = (-1)^k$.

<u>Def</u> $(T_n f)(z) = \dfrac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0\leq b<d} f\left(\dfrac{az+b}{d}\right)$ for $n \geq 1$ an integer

<u>Proposition</u> For any $n \geq 1$ T_n acts on modular forms and cusp forms.

We have linear maps: $T_n : M_k(q,\chi) \longrightarrow M_k(q,\chi)$

$T_n : S_k(q,\chi) \longrightarrow S_k(q,\chi)$.

We come back to this later, but to prove it we will need to re-interpret in terms of group theory.

Let $\Gamma = SL_2\mathbb{Z}$

Consider $M_n = \left\{ \gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, \det \gamma = n \right\}$

So $M_1 = \Gamma$.

Γ acts on the left and on the right by multiplication.

<u>Lemma</u> The set

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad = n, \ 0 \leq b < d \right\}$$

is a complete set of right coset representatives of M_n mod Γ. I.e.

$$M_n = \bigsqcup_{\rho \in \Delta_n} \Gamma\rho.$$

PROOF Let $r = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in M_n$

Then $\exists\, \delta, \gamma$, such that $\gamma a + \delta c = 0$ and $(\gamma, \delta) = 1$

Indeed, just take $\gamma = \frac{c}{(a,c)}$, $\delta = \frac{-a}{(a,c)}$

Hence $\exists\, \tau = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in SL_2\mathbb{Z}$ such that $\tau r = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$

with $a'd' = n$, $d' > 0$ (since if $d' < 0$, we can take $-\tau$ instead of τ).

Now $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} a' & b' + d'u \\ 0 & d' \end{pmatrix}$, so we pick u to make $0 \leq b' + d'u < d'$

Thus we have $\tau' = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\tau$ st $\tau' r \in \Delta_n$.

Thus $M_n = \bigcup_{\rho \in \Delta_n} \Gamma\rho$.

Still need to show the union is disjoint.

Let $\rho_1, \rho_2 \in \Delta_n$. Consider $\Gamma\rho_1$, $\Gamma\rho_2$, suppose they are not disjoint.

Calculate: $\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b + \beta d \\ \gamma a & \gamma b + \delta d \end{pmatrix}$

but $\gamma a = 0$, so $\gamma = 0$

$= \begin{pmatrix} \alpha a & \alpha b + \beta d \\ 0 & \delta d \end{pmatrix}$ $\quad a\alpha\delta d = n$

so $\alpha\delta = 1$, since $d > 0$

$\Rightarrow \alpha = \delta = 1$.

Since $0 \leq b + \beta d < d$, we get $\beta = 0$.

Then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $\rho_1 = \rho_2$. So if $\rho_1 \neq \rho_2$ then $\Gamma\rho_1$, $\Gamma\rho_2$ Disjoint

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Consequence: Given $\rho \in \Delta_n$, $\tau \in \Gamma$, there exists a unique $\tau'_n, \rho' \in \Delta_n$ such that $\boxed{\rho\tau = \tau'\rho'}$, i.e. We get a bijection: $SL_2\mathbb{Z} \times \Delta_n \xrightarrow{\sim} SL_2\mathbb{Z} \times \Delta_n$.

"Intertwining Relation".

We don't yet need to assume χ is a Dirichlet character.
For now, we only suppose it is a fcn $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$.

Let $\chi(\rho) = \bar{\chi}(a)$ if $\rho = \begin{pmatrix} a & * \\ * & * \end{pmatrix}$.

then $$T_n f = n^{k/2-1} \sum_{\rho \in \Delta_n} \bar{\chi}(\rho) f|_\rho \qquad \circledast$$
$$= \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right).$$

We begin by studying T_n on periodic fcns.
Now assume $\chi(1) = 1$, $\chi(-1) = (-1)^k$. ~~now~~

Let Γ_∞ = stabilizer of ∞ $= \left\{ \pm \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, u \in \mathbb{Z} \right\}$.

We study $f|_\tau = \chi(\tau) f$, $\tau \in \Gamma_\infty$, that is, f is periodic.

<u>Lemma</u> T_n maps periodic fcns to periodic fcns, that is,
$$T_n : M_k(\Gamma_\infty, \chi) \longrightarrow M_k(\Gamma_\infty, \chi).$$

<u>Proof</u> Explicitly, we have
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b+au-dv \\ 0 & d \end{pmatrix} \quad \text{i.e. } \rho\tau = \tau'\rho' \quad \tau, \tau' \in \Gamma_\infty.$$

If $\rho\tau = \tau'\rho'$ w/ $\tau \in \Gamma_\infty$, then $f|_{\rho\tau} = f|_{\tau'\rho'} = \chi(\tau') f|_{\rho'}$

Note: $\bar{\chi}(\rho)\chi(\tau') = \bar{\chi}(\rho')\chi(\tau)$ (direct check).

Then $$(T_n f)|_\tau = n^{k/2-1} \sum_{\rho \in \Delta_n} \bar{\chi}(\rho) f|_{\rho\tau} = n^{k/2-1} \sum_{\rho \in \Delta_n} \bar{\chi}(\rho)\chi(\tau') f|_{\rho'}$$
$\tau \in \Gamma_\infty$
$$= n^{k/2-1} \chi(\tau) \sum_{\rho \in \Delta_n} \bar{\chi}(\rho') f|_{\rho'}$$

Bijection:
$$\xrightarrow{\qquad} \quad = n^{k/2-1} \chi(\tau) \sum_{\rho' \in \Delta_n} \bar{\chi}(\rho') f|_{\rho'}$$
Given ρ, τ choose v, i.e. choose τ' such that
$0 \le b + au - dv < d$
$$= n^{k/2-1} \chi(\tau) \, T_n f.$$
So $\Gamma_\infty \times \Delta_n \xrightarrow{\sim} \Gamma_\infty \times \Delta_n$.

QED

Now let $f \in M_k(\Gamma_\infty, \chi)$, i.e. f is given by an abs. cvg Fourier Series:
$$f(z) = \sum_{m=0}^{\infty} a(m) e(mz).$$

Lemma We have $(T_n f)(z) = \sum_{m=0}^{\infty} a_n(m)\, e(mz)$

with $a_n(m) = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1} a\left(\frac{mn}{d^2}\right)$.

Proof We apply ⊛

$$(T_n f)(z) = \frac{1}{n} \sum_{m=0}^{\infty} a(m) \sum_{ad=n} \chi(a)\, a^k \sum_{b(d)} e\left(m\, \frac{az+b}{d}\right)$$

$$= \frac{1}{n} \sum_{m=0}^{\infty} a(m) \sum_{ad=n} \chi(a)\, a^k\, e\left(m \frac{az}{d}\right) \underbrace{\sum_{b(d)} e\left(\frac{mb}{d}\right)}_{= d\cdot\delta_{d\mid m}}.$$

let $l = m/d$

$$= \sum_{l=0}^{\infty} \sum_{ad=n} \chi(a)\, a^{k-1} a(dl)\, e(laz)$$

let $m = la$

re-use of the letter m for a different meaning...

$$= \sum_{m=0}^{\infty} \left(\sum_{\substack{da=n \\ al=m}} \chi(a)\, a^{k-1} a(dl) \right) e(mz)$$

Note

$$dl = \frac{ad.\, al}{a^2} = \frac{mn}{a^2} \implies = \sum_{m\geq 0} \left(\sum_{a \mid (m,n)} \chi(a)\, a^{k-1} a\left(\frac{mn}{a^2}\right)\right) e(mz).$$

Q.E.D.

We can deduce many consequences about Hecke operators from this:

COR $a_n(0) = \left(\sum_{d \mid n} \chi(d)\, d^{k-1} \right) a(0)$

and if $m,n \geq 1$ then $a_n(m) = a_m(n)$.

Now assume χ is completely multiplicative: $\chi(mn) = \chi(m)\chi(n)$ $\forall m,n \geq 1$.

Theorem $T_m T_n = \sum_{d \mid (m,n)} \chi(d)\, d^{k-1}\, T_{\frac{mn}{d^2}}$

PROOF By definition of T_n

$$mn\,T_mT_n = \sum_{\substack{a_1d_1=m\\a_2d_2=n}} \chi(a_1a_2)(a_1a_2)^k \sum_{\substack{b_1 \bmod d_1\\ b_2 \bmod d_2}} \underbrace{\Big|\begin{pmatrix} a_1 & b_1\\ 0 & d_1\end{pmatrix}\Big|\begin{pmatrix} a_2 & b_2\\ 0 & d_2\end{pmatrix}}_{=\,\big|\begin{pmatrix} a_1a_2 & a_1b_2+b_1d_2\\ 0 & d_1d_2\end{pmatrix}}$$

Note that $\delta=(a_1,d_2)$ divides each entry of this.

Scalars act trivially by slash operator, so we change vars:

$$a_1/\delta \to a_1 \qquad d_2/\delta \to d_2$$

$$mn\,T_mT_n = \sum_{\delta\mid(m,n)} \sum_{\substack{d_1a_1=m/\delta\\ a_2d_2=n/\delta\\ (a_1,d_2)=1}} \chi(a_1a_2\delta)(a_1a_2\delta)^k \sum_{\substack{b_1(d_1)\\ b_2(\delta d_2)}} \Big|\begin{pmatrix} a_1a_2 & a_1b_2+b_1d_2\\ 0 & d_1d_2\end{pmatrix}.$$

Now, $(a_1,d_2)=1$, so as b_1, b_2 run mod d_1, δd_2, $a_1b_2+b_1d_2$ represents each integer mod d_1d_2 exactly δ times.

Indeed,

$$\mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \xrightarrow[\text{bij}]{\sim} \mathbb{Z}/d_1d_2$$
$$(b_1,b_2) \longmapsto a_1b_2+b_1d_2,$$

with inverse $b \mapsto \left(\frac{b-a_1b_2}{d_2} \bmod d_1,\ b\bar{a}_1 \bmod d_2\right)$

Warning: NOT a homomorphism.

So

$$mn\,T_mT_n = \sum_{\delta\mid(m,n)} \chi(\delta)\,\delta^{k+1} \sum_{\substack{a_1d_1=m/\delta\\ a_2d_2=n/\delta\\ (a_1,d_2)=1}} \chi(a_1a_2)(a_1a_2)^k \sum_{b(d_1d_2)} \begin{pmatrix} a_1a_2 & b\\ 0 & d_1d_2\end{pmatrix}$$

Let $a=a_1a_2$, $d=d_1d_2$, so $\frac{mn}{\delta^2}=ad$

We have a bijection

$$\left\{a,d : ad=\frac{mn}{\delta^2}\right\} \xleftrightarrow{\ \sim\ } \left\{a_1,a_2,d_1,d_2 : \begin{matrix}a_1d_1=m/\delta\\ a_2d_2=n/\delta\end{matrix},\ (a_1,d_2)=1\right\}.$$

Indeed, $(a,d) \longmapsto a_1=\frac{m}{(m,\delta d)},\ a_2=\frac{a}{a_1},\ d_1=d/d_2,\ d_2=\frac{\delta d}{(m,\delta d)}$

$$(a_1a_2, d_1d_2) \longleftarrow\!\shortmid (a_1,a_2,d_1,d_2)$$

So:

$$mn T_m T_n = \sum_{\delta \mid (m,n)} \chi(\delta)\,\delta^{k+1} \sum_{ad = \frac{mn}{\delta^2}} \chi(a)\, a^k \sum_{b(d)} 1 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

So $T_m T_n = \sum_{\delta \mid (m,n)} \chi(\delta)\,\delta^{k-1}\, T_{\frac{mn}{\delta^2}}$, as was to be shown QED.

Corollary For all $m,n \geq 1$ $T_n T_m = T_n T_m$.

Inversion: Let $\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free} \\ 0 & \text{else} \end{cases}$

μ is called the möbius function. Multiplicative: $\mu(n)\mu(m) = \mu(nm)$ if $(m,n)=1$.

Also: $\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{else.} \end{cases}$ "Möbius inversion"

Indeed: $\sum_{d \mid n} \mu(d) = \prod_{p \mid n} (1 + \mu(p)) = \begin{cases} 1 & \text{if product is empty} \\ 0 & \text{else.} \end{cases}$

Then $T_{mn} = \sum_{d \mid (m,n)} \mu(d)\chi(d)\, d^{k-1}\, T_{m/d}\, T_{n/d}$

PROOF $\text{RHS} = \sum_{d \mid (n,m)} \mu(d)\chi(d)\, d^{k-1} \sum_{\delta \mid (\frac{m}{d}, \frac{n}{d})} \chi(\delta)\, \delta^{k-1}\, T_{\frac{mn}{d^2\delta^2}}$

let $l = \delta d$ $= \sum_{l \mid (m,n)} \chi(l)\, l^{k-1} \sum_{d \mid l} \mu(d)\, T_{\frac{mn}{l^2}} = \sum_{l \mid (m,n)} \chi(l)\, l^{k-1}\, T_{\frac{mn}{l^2}}\, \delta_{l=1}$

$= T_{mn}$. QED.

In particular: $T_{mn} = T_m T_n$ if $(m,n)=1$.

It suffices to study T_n on prime powers. We have if $m = p^\nu$, $n = p$ that

$$T_{p^{\nu+1}} = T_p T_{p^\nu} - \chi(p)\, p^{k-1}\, T_{p^{\nu-1}}$$

$$\sum_{\nu \geq 0} T_{p^\nu} X^\nu = \frac{1}{1 - T_p X + \chi(p) p^{k-1} X^2}.$$

Let $\mathbb{T}$ the algebra generated by all T_n over $\mathbb{C}$. It is called the Hecke alg.

$\mathbb{T}$ is generated by T_p, ~~am~~ p prime, and is a commutative algebra.

Now we prove if $\Gamma = SL_2\mathbb{Z}$

Thm The Hecke operators maps a modular form to a modular form and a cusp form to a cusp form. That is, we have linear maps

$$T_n : M_k(SL_2\mathbb{Z}) \to M_k(SL_2\mathbb{Z})$$
$$T_n : S_k(SL_2\mathbb{Z}) \to S_k(SL_2\mathbb{Z})$$

Proof By today's first lemma, given ρ, τ, $\exists!\ \tau', \rho'$ st $\rho\tau = \tau'\rho'$

I.e. we have a bijection $\Delta_n \times SL_2\mathbb{Z} \to \Delta_n \times SL_2\mathbb{Z}$.

So $(T_nf)|_\tau = n^{k/2-1} \sum_{\rho\in\Delta_n} f|_{\rho\tau} = n^{k/2-1} \sum_{\rho'\in\Delta_n} f|_{\rho'} = T_nf$.

If $F(z) = y^{k/2}|f(z)|$ is bounded on $\mathcal{H}$, then $y^{k/2}|f|_\rho(z)|$ is also bounded

$\left[f|_\rho = n^{k/2} \cdot d^{-k} \cdot f\left(\frac{az+b}{d}\right) \right]$, so $y^{k/2}|T_nf(z)|$ is also bounded.

Therefore $T_n : S_k(1) \to S_k(1)$
