We showed: If $\chi : \mathbb{Z} \to \mathbb{C}$ is any function satisfying $\chi(mn) = \chi(n)\chi(m)$ $mn \geq 1$ with $\chi(-1) = (-1)^k$ Then

Theorem $$T_m T_n = \sum_{d \mid (m,n)} \chi(d) d^{k-1} T_{\frac{mn}{d^2}}$$

COR For all $m,n \geq 1$, $T_m T_n = T_n T_m$

For $(n,m) = 1$, $T_m T_n = T_{mn}$

Inversion: let $\mu(n) = \begin{cases} 1 & \text{if } n \text{ is sq-free} \\ 0 & \text{else} \end{cases}$

μ is called the Möbius function. Multiplicative: If $(m,n)=1$ then $\mu(n)\mu(m) = \mu(nm)$

We have $$\sum_{d \mid n} \mu(d) = \prod_{p \mid n} (1 + \mu(p) + \mu(p^2) + \cdots)$$
$$= \prod_{p \mid n} (1+\mu(p)) = \begin{cases} 1 & \text{if product is empty} \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{else} \end{cases}$$

"Möbius inversion"

So $$\sum_{d \mid (m,n)} \mu(d) \chi(d) d^{k-1} T_{m/d} T_{n/d} = \sum_{d \mid (m,n)} \mu(d) \chi(d) d^{k-1} \sum_{\delta \mid (\frac{m}{d}, \frac{n}{d})} \chi(\delta) \delta^{k-1} T_{\frac{mn}{d^2 \delta^2}}$$

Let $l = \delta d$
$$= \sum_{l \mid (m,n)} \chi(l) l^{k-1} T_{\frac{mn}{l^2}} \sum_{d \mid l} \mu(d) = T_{mn}.$$

So, it suffices to study Hecke operators on prime powers:

Let $m = p^v$, $n = p$, Then $T_{p^{v+1}} = T_{p^v} T_p - \chi(p) p^{k-1} T_{p^{v-1}}$

Also: $$\sum_{v \geq 0} T_{p^v} X^v = \frac{1}{1 - T_p X + \chi(p) p^{k-1} X^2}$$

The algebra $\mathbb{T}$ generated by all T_n over $\mathbb{C}$ is called the Hecke Algebra

It is generated by T_p for all p prime and is commutative.

Let $\Gamma = SL_2\mathbb{Z}$

<u>Theorem</u> The Hecke operators map modular forms to modular forms and cusp forms to cusp forms. That is:

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$$T_n : M_k(\Gamma) \longrightarrow M_k(\Gamma)$$
$$T_n : S_k(\Gamma) \longrightarrow S_k(\Gamma).$$

<u>PROOF</u> By the lemma from last class, the intertwining relation $\rho\tau = \tau'\rho'$ $\rho,\rho' \in \Delta_n,\ \tau,\tau' \in \Gamma$ defines a bijection $\Delta_n \times \Gamma \xrightarrow{\sim} \Delta_n \times \Gamma$, so:

$$(T_n f)\big|_\tau = n^{k/2-1} \sum_{\rho \in \Delta_n} f\big|_{\rho\tau} = n^{k/2-1} \sum_{\rho' \in \Delta_n} f\big|_{\rho'} = T_n f.$$

If $F(z) = y^{k/2}|f(z)|$ is bounded on $\mathcal{H}$, then since

$$y^{k/2}\,|f|_\rho(z)| = y^{k/2} n^{k/2} d^{-k} \left|f\left(\frac{az+b}{d}\right)\right|$$
$$= \left(\frac{a}{d}\right)^{k/2} y^{k/2} \left|f\left(\frac{az+b}{d}\right)\right| = F\left(\frac{az+b}{d}\right)$$

$F(z)$ bounded on $\mathcal{H} \Rightarrow F\left(\frac{az+b}{d}\right) = y^{k/2}|f|_\rho(z)|$ bounded on $\mathcal{H}$.

$\Rightarrow y^{k/2}|T_n f(z)|$ bounded on $\mathcal{H}$. $\Rightarrow T_n f \in S_k(\Gamma)$ if $f \in S_k(\Gamma)$.

Digression on Poincaré Series and Eisenstein series.

Now Γ is a congruence subgroup. For each cusp of Γ, there is an Eisenstein series and Poincaré series attached to that cusp. These are the <u>basic</u> examples of modular forms.

We only consider $\Gamma = \Gamma_0(q)$, and Eis & Poincaré series at the cusp at ∞ for simplicity.

For the more general case, see Iwaniec Ch 3.

Let $k > 2$ and χ mod q a Dirichlet char. Let $E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \overline{\chi}(q)\, j(\gamma, z)^{-k}$ $k \in \mathbb{Z}$

Here, recall we have $j(\gamma,z) = cz+d$.

$E_k(z)$ converges absolutely since $k>2$.

By construction $E_k \in \mathcal{F}_k(q,\chi)$, by convergence, it is holomorphic.

We will check that it has rapid decay at ∞ (the calculation at other cusps is similar)

More generally, for $m \geq 0$ $m \in \mathbb{Z}$, we define the mth Poincaré Series: (of wt k, level q, char χ) at ∞

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \bar{\chi}(\gamma) j(\gamma,z)^{-k} e(m\gamma z).$$

This converges absolutely if $k>2$, and is in $\mathcal{F}_k(q,\chi)$.

In fact, $E_k = P_0$, and $P_m \in M_k(q,\chi)$ $m \geq 0$
$P_m \in S_k(q,\chi)$ $m \geq 1$

The growth condition follows from the following Lemma at ∞ (other cusps are treated similarly, see IW ch 3)

<u>Lemma</u> The Poincaré series P_m has Fourier expansion

$$P_m(z) = \delta(m,0) + \sum_{n\geq 1} p(m,n)\, e(nz)$$

with $$p(0,n) = \left(\frac{2\pi}{i}\right) \frac{n^{k-1}}{\Gamma(k)} \sum_{\substack{c>0 \\ c\equiv 0(q)}} c^{-k} S_\chi(0,n,c)$$

and if $m\geq 1$

$$p(m,n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \left(\delta(m,n) + 2\pi i \sum_{\substack{c>0 \\ c\equiv 0(q)}} \frac{S_\chi(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right).$$

Here $S_\chi(m,n,c)$ is the Kloosterman Sum

$$S_\chi(m,n,c) = \sum_{d \bmod c}^{*} \chi(d)\, e\left(\frac{md+n\bar{d}}{c}\right)$$, where $*$ means $(d,c)=1$, $\bar{d} = d$ inverse mod c.

J_{k-1} is the Bessel function of order $k-1$

Theorem If $\chi =$ trivial, write $S(m,n,c)$ for the Kloosterman sum.
(Weil bound for the Kloosterman sum)

Let a,b,c be integers, $c>0$

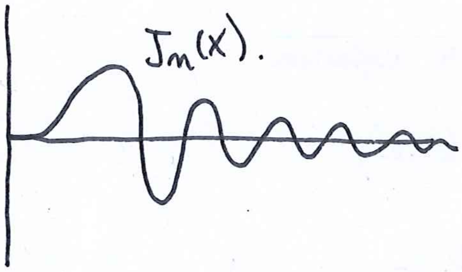
$$|S(a,b;c)| \leq d(c)(a,b,c)^{1/2} c^{1/2}$$

The Proof is via the Riemann hypothesis for the zeta function of curves over finite fields (proved by Weil in the 1940s).

[for Varieties, thm of Deligne from 1974 $\Rightarrow$ optimal bound on Fourier coeffs....]

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ni\theta + iz\sin\theta}\, d\theta, \quad n = 0,1,2,\dots$$

$$= \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} \frac{\Gamma(-t)}{\Gamma(\nu+t+1)}\left(\frac{x}{2}\right)^{\nu+2t} dt \quad \text{if } \operatorname{Re}(\nu)>0 \text{ etc.}$$



Satisfies:

$$J_{k-1}(x) \ll \min\left(1, \frac{x}{k}\right) k^{-1/3}$$

$$J_\nu(x) \ll x^{-\frac{1}{4}}\left(|x-\nu| + \nu^{1/3}\right)^{-1/4}$$

$$\ll \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu \quad \dots$$

Proof We have from the Bruhat decomposition (you saw this in exercises)

$$\Gamma_0(q) = \Gamma_\infty \sqcup \left(\bigsqcup_{\substack{c>0 \\ c\equiv 0(q)}} \bigsqcup_{\substack{d(c) \\ (c,d)=1}} \Gamma_\infty \begin{pmatrix} a & b \\ c & d\end{pmatrix} \Gamma_\infty \right),$$

where a,b are any integers such that $ad-bc=1$.

$$\text{Then } P_m(z) = \cancel{\sum_{n\in\mathbb{Z}}} = e(mz) + \sum_{\substack{c>0 \\ c\equiv 0(q)}} \sum_{d(c)}^{*} \bar{\chi}(d)\, I(c,d;z)$$

where

$$I(c,d;z) = \sum_{n\in\mathbb{Z}} \left(c(z+n)+d\right)^{-k} e\left(m\left(\frac{a}{c} - \frac{1}{c(c(z+n)+d)}\right)\right)$$

Since $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix} = \begin{pmatrix} a & an+b \\ c & cn+d\end{pmatrix}$ and $\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$.

By Poisson Summation

$$I(c,d,z) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \left(c(z+v)+d\right)^{-k} e\left(\frac{am}{c} - \frac{m}{c(c(z+v)+d)} - nv\right) dv$$

$$= \sum_{n\in\mathbb{Z}} e\left(\frac{am}{c}+\frac{nd}{c}\right)\left(\int_{-\infty+iy}^{\infty+iy} (cv)^{-k} e\left(\frac{-m}{c^2 v} - nv\right) dv\right) e(nz) \qquad \begin{matrix} cu = c(z+v)+d \\ u - z - \frac{d}{c} = v \end{matrix}$$

If $n \le 0$ then $\int (cv)^{-k} e\left(\frac{-m}{c^2 v} - nv\right) dv = 0$

by shifting the contour up.

So: $$P_m(z) = \sum_{n\geq 1} \left[\sum_{\substack{c\equiv 0 (q)\\ c>0}} \int_{-\infty+iy}^{\infty+iy} (cv)^{-k} e\left(\frac{-m}{c^2 v} - nv\right) dv \left(\sum_{d(c)}^{*} e\left(\frac{nd}{c}\right)\bar{\chi}(d)\, e\left(\frac{am}{c}\right)\right)\right] \cdot e(nz)$$

$$+\ e(mz)$$

We have $$\int_{\operatorname{Im} v = \operatorname{Im} z > 0} \frac{e\left(\frac{-m}{c^2 v} - nv\right)}{(cv)^{k-1}} \frac{dv}{v} = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$ if $m \geq 1$

which follows from the second formula... "Mellin-Barnes integral representation"

and $$\frac{-1}{\Gamma(z)} = \frac{1}{2\pi i}\int (-t)^{-z} e^{-t} dt.$$

Cor. If $m \geq 1$ then $P_m \in S_k(q,\chi)$.

There are ∞-ly many Poincaré series, but $S_k(q,\chi)$ is finite-dimensional.

<u>Lemma</u> Let $f \in M_k(q,\chi)$, and write $f(z) = \sum_{n\geq 0} a(n) e(nz)$.

For any $m \geq 1$ $$\langle f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m)$$

<u>Corollary</u> The Poincaré series span $S_k(q,\chi)$.

Indeed, if f were orthogonal to all P_m, then all of its Fourier coeffs would be zero. The space spanned by Poincaré series is <u>closed</u> since $S_k(q,\chi)$ is F.D.

<u>Proof of Lemma</u> By defn of P_m

$$\langle f, P_m\rangle = \int_{\mathcal{F}_{\Gamma_0(q)}} f(z)\Big(\overline{\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(q)} \chi(\gamma)\, j(\gamma,z)^{-k}\, e(m\gamma z)}\Big) y^k \frac{dx\,dy}{y^2}$$

$$= \int_{\mathcal{F}_{\Gamma_0(q)}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(q)} (\operatorname{Im}\gamma z)^k f(\gamma z)\overline{e(m\gamma z)}\,\frac{dx\,dy}{y^2}$$

by modularity of f and $\operatorname{Im}\gamma z = \dfrac{\operatorname{Im} z}{|cz+d|^2}$

"Unfolding"
$$= \int_0^1\int_0^\infty y^{k-2} f(z)\, e(-m\bar z)\,dx\,dy$$

$$= \sum_{n\geq 0} a_f(n)\int_0^\infty y^{k-2} e^{-2\pi(m+n)y}\,dy \underbrace{\int_0^1 e((n-m)x)\,dx}_{\begin{cases}1 & \text{if } m=n\\ 0 & \text{else}\end{cases}}$$

$$= \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, a_f(m).$$

Let $\mathcal{F}$ be any orthonormal basis of $S_k(q,\chi)$

Write $P_m = \sum_{f\in\mathcal{F}} \langle f, P_m\rangle f$, take $\langle\cdot, P_n\rangle$ of both sides and use previous two results. $\Rightarrow$

<u>Theorem</u> (Petersson Trace Formula)

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f\in\mathcal{F}} a_f(n)\overline{a_f(m)} = \delta(m,n) + 2\pi i^{-k}\sum_{\substack{c>0\\ c\equiv 0(q)}} \frac{S_\chi(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

"Almost orthogonality of Fourier coefficients".

Action of Hecke operators on Poincaré Series:

<u>Theorem</u> $T_n P_m = \sum_{d\mid(m,n)} \left(n/d\right)^{k-1} P_{\frac{mn}{d^2}}$ P on $SL_2\mathbb{Z}$, trivial char.

PROOF ~~We have~~

Note that since we are on $SL_2\mathbb{Z}$, $f|_\rho$, $\rho\in\Delta_n$ ~~does not depend~~ is the same for any elt of $\Gamma\rho$.

So $T_n f = n^{k/2-1}\sum_{\rho\in\Gamma\backslash M_n} f|_\rho$ if $f\in M_k(SL_2\mathbb{Z})$.

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Then
$$(T_n P_m)(z) = n^{k-1}\sum_{\rho\in\Gamma\backslash M_n} P_m|_\rho$$
$$= n^{k-1}\sum_{g\in\Gamma_\infty\backslash M_n} j(g,z)^{-k} e(mgz)$$
by unfolding and the cocycle condition.
$$= n^{k-1}\sum_{\rho\in\Gamma\backslash M_n}\sum_{\tau\in\Gamma_\infty\backslash\Gamma} j(\rho\tau,z)^{-k} e(m\rho\tau z)$$
$$= n^{k-1}\sum_{ad=n} d^{-k}\sum_{b(d)}\sum_{\tau\in\Gamma_\infty\backslash\Gamma} j_\tau(z)^{-k} e\left(m\frac{a\tau z+b}{d}\right)$$
$$= n^{k-1}\sum_{\substack{ad=n\\ d\mid m}} d^{1-k}\sum_{\tau\in\Gamma_\infty\backslash\Gamma} j_\tau(z)^{-k} e\left(\frac{am}{d}\tau z\right)$$
$$= \sum_{d\mid(m,n)}\left(\frac{n}{d}\right)^{k-1} P_{\frac{mn}{d^2}}$$

If $m=0$, $\Rightarrow$ $E_k = P_0$ is an eigenfunction of all T_n

with $\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1}$ as eigenvalue

$$T_n E_k = \sigma_{k-1}(n) E_k.$$

COR For $m,n\geq 1$

$$m^{k-1} T_n P_m = n^{k-1} T_m P_n$$

We also have $\frac{(4\pi m)^{k-1}}{\Gamma(k-1)}\langle T_n f, P_m\rangle = a_n(m) = a_m(n)$

So $m^{k-1}\langle T_n f, P_m\rangle = n^{k-1}\langle T_m f, P_n\rangle$