Recall $P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \bar{\chi}(\gamma)\, e(m\gamma z)\, j(\gamma,z)^{-k}$

Fourier series:

$$P_m(z) = \sum_{n \geq 0} p(m,n)\, e(nz)$$

Last time, we computed $p(m,n)$ explicitly in terms of Kloosterman sums and Bessel functions.

COR For $m \geq 1$ $P_m \in S_k(q,\chi)$.

Lemma Let $f \in M_k(q,\chi)$, write $f(z) = \sum_{n\geq 0} a_f(n)\, e(nz)$.

For any $m \geq 1$ $\langle f, P_m \rangle = \dfrac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, a_f(m)$.

Warning: Here $\langle\, ,\rangle$ is $\int_{\mathcal{D}_\Gamma} f\bar{g}\, y^k \frac{dxdy}{y^2}$, i.e. un-normalized $\langle\, ,\rangle_{\Gamma_0(q)}$.

COR The Poincaré Series span $S_k(q,\chi)$

PF The Poincaré series span a closed subspace of $S_k(q,\chi)$ by finite dimension. Suppose this space isn't all $S_k(q,\chi)$. Then there exists f orthogonal to the closed space spanned by the P_m. But by the lemma, such an f has all its Fourier coefficients equal to 0. Hence $f \equiv 0$.

PROOF OF LEMMA

By def: $\langle f, P_m\rangle = \int_{\mathcal{D}_{\Gamma_0(q)}} f(z) \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \chi(\gamma)\, \overline{j(\gamma,z)^{-k}\, e(m\gamma z)} \right) y^k \frac{dxdy}{y^2}$

$$= \int_{\mathcal{D}_{\Gamma_0(q)}} \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} (\operatorname{Im} \gamma z)^k f(\gamma z)\, \overline{e(m\gamma z)} \right) \frac{dxdy}{y^2}$$

using $f(z) = \chi(\gamma) j(\gamma,z)^{-k} f(\gamma z)$, $y = |j(\gamma,z)|^2 \operatorname{Im}\gamma z$

"Unfolding technique"

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$$= \int_0^1 \int_0^\infty y^{k-2} f(z)\, e(-m\bar{z})\, dx\,dy$$

$$= \sum_{n\geq 0} a_f(n) \int_0^\infty y^{k-2} e^{-2\pi(m+n)y} dy \underbrace{\int_0^1 e((n-m)x)\,dx}_{= \begin{cases} 1 & \text{if } n=m \\ 0 & \text{else.} \end{cases}}$$

$$= a_f(m) \int_0^\infty y^{k-2} e^{-4\pi m y} dy = a_f(m) \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}$$

Q.E.D.

Let $\mathcal{F}$ be any orthonormal basis for $S_k(q,\chi)$

Write $P_m = \sum_{f\in\mathcal{F}} \langle P_m, f\rangle f$

Apply $\langle \cdot, P_n\rangle$ to both sides:

$$\langle P_m, P_n\rangle = \sum_{f\in\mathcal{F}} \langle P_m, f\rangle \langle f, P_n\rangle$$

By the lemma:

$$p(m,n) = \sum_{f\in\mathcal{F}} \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \overline{a_f(m)}\, a_f(n)$$

From last week's explicit calculation of $p(m,n)$ we have:

Theorem (Petersson trace formula)

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f\in\mathcal{F}} a_f(n)\overline{a_f(m)} = \delta(m,n) + 2\pi i^{-k} \sum_{\substack{c>0 \\ c\equiv 0\,(q)}} \frac{S_\chi(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

$\mathcal{F}$ is any O.N. basis. "Almost orthogonality" as $q+k\to\infty$

Action of Hecke operators on Poincaré series:

Theorem If $\Gamma = SL_2\mathbb{Z}$, $k>2$, $n\geq 1$, $m \geq 0$ then

$$T_n P_m = \sum_{d \mid (m,n)} \left(\frac{n}{d}\right)^{k-1} P_{\frac{mn}{d^2}}$$

Proof Recall for $SL_2\mathbb{Z} = \Gamma$ we have for $f \in M_k(\Gamma)$

$$T_n f = n^{k/2-1} \sum_{\rho \in \Gamma \backslash M_n} f|_\rho. \qquad \text{(this is well-defined).}$$

So

$$\begin{aligned}
(T_n P_m)(z) &= n^{k/2-1} \sum_{\rho \in \Gamma\backslash M_n} P_m|_\rho \\
&= n^{k-1} \sum_{\rho \in \Gamma\backslash M_n} \sum_{\tau \in \Gamma_\infty \backslash \Gamma} j(\rho,z)^{-k} j(\tau,\rho z)^{-k} e(m\tau\rho z) \\
? \quad &= n^{k-1} \sum_{\rho \in \Gamma\backslash M_n} \sum_{\tau \in \Gamma_\infty\backslash\Gamma} j(\tau\rho,z)^{-k} e(m\tau\rho z) \\
\text{Intertwining rel'n} \quad &= n^{k-1} \sum_{\rho' \in M_n / \Gamma} \sum_{\tau \in \Gamma_\infty \backslash \Gamma} j(\rho'\tau',z)^{-k} e(m\rho'\tau' z). \\
&= n^{k-1} \sum_{da=n} d^{-k} \sum_{b \bmod d} \sum_{\tau} j(\tau,z)^{-k} e\left(m\left(\frac{a\tau z+b}{d}\right)\right) \\
&= n^{k-1} \sum_{\substack{ad=n \\ d\mid m}} d^{1-k} \sum_{\tau} j(\tau,z)^{-k} e\left(\frac{ma}{d}\tau z\right) \\
&= \sum_{d\mid(m,n)} \left(\frac{n}{d}\right)^{k-1} P_{\frac{mn}{d^2}}.
\end{aligned}$$

Cor For $m,n \geq 1$

$$m^{k-1} T_n P_m = n^{k-1} T_m P_n$$

Let $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, Then $T_n E_k = \sigma_{k-1}(n) E_k \quad \forall n \geq 1$.

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We also have by the lemma:

Recall!

$$\langle T_n f, P_m \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_n(m) = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_m(n)$$

$\left(\text{Writing } T_n f = \sum_{m \geq 0} a_n(m) e(mz)\right)$.

<u>Prop</u> For $m, n \geq 1$, $f \in M_k(\Gamma)$

$$m^{k-1} \langle T_n f, P_m \rangle = n^{k-1} \langle T_m f, P_n \rangle.$$

Let V a finite dim'l inner product space $/ \mathbb{C}$.

For any linear transformation $T: V \to V$, we define its adjoint transform $T^*: V \to V$ by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \qquad \forall x, y \in V.$$

If $TT^* = T^*T$ then T is called normal

If $T = T^*$ then T is called self-adjoint.

<u>Theorem</u> The Hecke operators are self-adjoint on $S_k(\Gamma)$.

That is: $\langle T_n f, g \rangle = \langle f, T_n g \rangle \quad \forall f, g \in S_k(\Gamma)$.

<u>Proof</u> Since $S_k(\Gamma)$ is spanned by Poincaré series, it suffices to check the formula on Poincaré series.

We compute: $(ml)^{k-1} \langle T_n P_m, P_\ell \rangle \underset{\text{COR}}{=} (\ell n)^{k-1} \langle T_m P_n, P_\ell \rangle$

$\underset{\text{Prop}}{=} (mn)^{k-1} \langle T_\ell P_n, P_m \rangle = (m\ell)^{k-1} \langle T_n P_\ell, P_m \rangle$

Since F.C. and inner prod are $\in \mathbb{R}$. QED.

Recall

~~Lem~~ **PROP** Let V a finite-dim'l inner product space $/\mathbb{C}$, and let $\mathcal{S}$ be a commuting family of normal operators $T: V \to V$. Then there exists an orthonormal basis $\mathcal{B}$ of V which consists of common eigenvalues of all the operators $T \in \mathcal{S}$.

Theorem There exists an orthonormal basis $\mathcal{F}$ of $S_k(\Gamma)$ where $\Gamma = SL_2\mathbb{Z}$, which consists of eigenfunctions of all the Hecke operators T_n.

Consequences:

We say $f \in S_k(\Gamma)$ is a Hecke eigenform if $\forall n = 1, 2, 3, \ldots$

$$T_n f = \lambda(n) f$$

Suppose $f(z) = \sum_{m=1}^{\infty} a(m)\, e(mz)$ is a Hecke eigenform.

We write $T_n f = \sum_{m=1}^{\infty} a_n(m)\, e(mz)$

we showed: $a_n(m) = \sum_{d \mid (m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right)$

thus $\lambda(n) a(m) = \sum_{d \mid (m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right)$

In particular: $\boxed{a(n) = \lambda(n) a(1)}$

So we must have $a(1) \neq 0$ otherwise $f \equiv 0$ since all F.C. vanish.

$\boxed{a(n) \text{ is proportional to } \lambda(n) \quad \forall n \geq 1}$

Two normalizations are common in litterature:

(1) "L^2-normalized" $f \in \mathcal{F}$ is an orthonormal basis of Hecke eigenforms. In this case $\langle f, f \rangle = 1$, but $a(1)$ varies.

② "Hecke Normalization" $f \in \mathcal{G}$ is an orthogonal (not normal) basis of Hecke eigenforms such that $a_f(1) = 1 \quad \forall f \in \mathcal{G}$. τ

In this case $\langle f, f \rangle$ varies.

We call f in the latter case Primitive forms.

The Fourier coefficients of a primitive form are equal to its Hecke eigenvalues.

Example $\quad q = e(z)$

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

is the unique primitive form in $S_{12}(\Gamma)$.

$$T_n \Delta = \tau(n) \Delta$$

Therefore: $$\tau(m)\tau(n) = \sum_{d \mid (m,n)} d^{k-1} \tau\left(\frac{mn}{d^2}\right)$$

Proving Ramanujan's Conjecture.

Hecke Theory for $S_k(q, \chi)$ — Most interesting examples happen here.

We only sketch the proofs – they are similar to the $SL_2\mathbb{Z}$ case.

$\rho \in SL_2\mathbb{Z}$, $\chi(\rho) = 0 \quad \forall \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $(a, q) = 1$.

So we make the following modifications to the theory:

Let $$\Delta_n^q = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_n : (a,q) = 1 \right\}$$

Thus $$T_n f = n^{k/2-1} \sum_{\rho \in \Delta_n} \overline{\chi}(\rho) f|_\rho = n^{k/2-1} \sum_{\rho \in \Delta_n^q} \overline{\chi}(\rho) f|_\rho = T_n^{\chi} f.$$

Check: For $\rho \in \Delta_n^q$, $\tau \in \Gamma_0(q)$ there exists unique $\tau' \in \Gamma_0(q)$, $\rho' \in \Delta_n^q$

such that $\rho\tau = \tau'\rho'$. (We can do this by an explicit matrix computation: Exercise)

Thus we have a bijection $\Delta_n^q \times \Gamma_0(q) \longleftrightarrow \Gamma_0(q) \times \Delta_n^q$

For such ρ, τ we also have $\chi(\rho)\chi(\tau) = \chi(\tau')\chi(\rho')$
(another exercise, or follows from the same calculation)

We suppose also $\chi(-1) = (-1)^k$, otherwise $S_k(\Gamma_0(q), \chi) = \{0\}$.

Theorem The Hecke operator T_n^χ maps modular forms to modular forms and cusp forms to cusp forms.

$$T_n^\chi : M_k(q,\chi) \to M_k(q,\chi)$$
$$T_n^\chi : S_k(q,\chi) \to S_k(q,\chi)$$

Proof $\bar\chi(\rho) f|_{\rho\tau} = \bar\chi(\rho) f|_{\tau'\rho'} = \bar\chi(\rho)\chi(\tau') f|_{\rho'}$
$$= \chi(\tau)\bar\chi(\rho') f|_{\rho'}$$

Thus $(T_n f)|_\tau = \chi(\tau) T_n f \quad \forall\, \tau \in \Gamma_0(q)$

As we saw before, $|_\rho$, $\rho \in \Delta_n^q$ preserves boundedness ~~at ∞~~ of $F(z)$.

We already showed for the larger space $M_k(\Gamma_{\infty}, \chi)$ that

$$T_m T_n = \sum_{d|(m,n)} \chi(d)\, d^{k-1}\, T_{\frac{mn}{d^2}},$$

So this relation also holds for $M_k(q,\chi)$.

Thus the Hecke algebra $\mathbb{T}$ is commutative, generated by T_p^χ

$$T_p^\chi = \frac{1}{p} \sum_{b(p)} |_{\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}} + \chi(p)\, p^{k-1} |_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}$$

If $p | q$ this is $T_p = \frac{1}{p} \sum_{b(p)} |_{\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}}$.