Hecke Theory for $S_k(q,\chi)$

Let $\Delta_n^q = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta_n : (a,q)=1 \right\}$.

$$T_n = T_n^\chi = n^{k/2-1} \sum_{\rho \in \Delta_n} \bar\chi(\rho) f|_\rho = n^{k/2-1} \sum_{\rho \in \Delta_n^q} \bar\chi(q) f|_\rho.$$

Exercise: Given $\rho \in \Delta_n^q$, $\tau \in \Gamma_0(q)$ there exists unique $\tau' \in \Gamma_0(q)$, $\rho' \in \Gamma_0(q)$ such that $\rho\tau = \tau'\rho'$.

Thus we have a bijection $\Delta_n^q \times \Gamma_0(q) \longrightarrow \Gamma_0(q) \times \Delta_n^q$.

Exercise: For such ρ, τ, we also have $\chi(\rho)\chi(\tau) = \chi(\tau')\chi(\rho')$

Suppose also that $\chi(-1) = (-1)^k$ (else $S_k(q,\chi) = \{0\}$)

Theorem The Hecke operators T_n^χ map modular forms to modular forms and cusp forms to cusp forms.

$$T_n^\chi : M_k(q,\chi) \longrightarrow M_k(q,\chi)$$
$$T_n^\chi : S_k(q,\chi) \longrightarrow S_k(q,\chi).$$

Proof $\bar\chi(\rho) f|_{\rho\tau} = \bar\chi(\rho) f|_{\rho'\tau'} = \bar\chi(\rho)\chi(\tau') f|_{\rho'}$

$$= \chi(\tau)\bar\chi(\rho') f|_{\rho'}$$

So $(T_n^\chi f)\big|_\tau = n^{k/2-1} \sum_{\rho \in \Delta_n^q} \bar\chi(q) f|_{\rho\tau} = \chi(\tau) n^{k/2-1} \sum_{\rho' \in \Delta_n^q} \bar\chi(\rho') f|_{\rho'}$

$= \chi(\tau) T_n^\chi f.$

As we saw before, $|_p$, $p \in \Delta_n^q$ preserves boundedness of $F(z)$.

We already showed that for the larger space $M_k(\Gamma_\infty, \chi)$ that

$$T_m T_n = \sum_{d|(m,n)} \chi(d)\, d^{k-1}\, T_{\frac{mn}{d^2}}$$

So this also holds for $M_k(\Gamma_0(q), \chi)$.

So $\mathbb{T}$ is commutative, generated by T_p^χ

$$T_p^\chi = \frac{1}{p} \sum_{b(q)} \Big|\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} + \chi(p) p^{k-1} \Big| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

If $p|q$ $T_p^\chi = \frac{1}{p} \sum_{b(p)} \Big| \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$ and $(T_{p^\nu}) = (T_p)^\nu$

So T_p^χ, $p|q$ behave very differently from the case $p \nmid q$.

Sadly, the previous proof for the existance of the orthonormal basis doesn't hold, since $T_n P_m$ only has the correct symmetry when $(nm,q)=1$

For example PROP If $m,n \geq 1$, $(mn,q)=1$ then

$$\chi(m) m^{k-1} T_n P_m = \chi(n) n^{k-1} T_m P_n$$

So we can only conclude that

$$\langle T_n f, g \rangle = \chi(n) \langle f, T_n g \rangle \text{ if } f,g \in \operatorname{Span}_{(m,q)=1} P_m.$$

But there's no reason that this spans the whole space.

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Alternate proof: Use normalized inner product:

$$\langle f,g\rangle = \frac{1}{[\underset{\Gamma_0(q)}{\cancel{SL_2(\mathbb{Z})}}:\Gamma]}\int_{\mathcal{D}_\Gamma} f(z)\overline{g(z)}\, y^k \,\frac{dx\,dy}{y^2}$$

Note that for all $\sigma \in GL_2^+(\mathbb{R})$, we have

$$f|_\sigma(z)\,\overline{g|_\sigma(z)}\; y^k\, d\mu(z) = f(\sigma z)\overline{g(\sigma z)}\,(\operatorname{Im}\sigma z)^k\, d\mu(\sigma z)$$

We choose Γ sufficiently small, but of finite index so that χ is trivial on Γ. E.g. $\Gamma_1(q) \subseteq \Gamma_0(q)$ suffices.

Then choose Γ' smaller but of finite index so that

$$|_\sigma : S_k(q,\chi) \longrightarrow S_k(\Gamma') \qquad (\text{for } \sigma \in GL_2^+(\mathbb{Q}))$$

For example, if $\sigma \in M_n$, then take $\Gamma' = \Gamma(nq)$ (exercise!)

PROP Let $\sigma \in GL_2^+(\mathbb{Q})$

$|_\sigma$ is an isometry of $S_k(q,\chi)$ with respect to the normalized Petersson inner product:

$$\langle f|_\sigma, g|_\sigma\rangle = \langle f,g\rangle \quad \text{for all } f,g \in S_k(q,\chi).$$

PROOF Let Γ' be a sufficiently small subgroup of finite index so that $|_\sigma : S_k(q,\chi) \longrightarrow S_k(\Gamma')$

Let $\mathcal{D}_{\Gamma'}$ a fundamental domain. Then

$$\langle f|_\sigma, g|_\sigma\rangle = [\Gamma_0(q):\Gamma']^{-1}\int_{\mathcal{D}_{\Gamma'}} f|_\sigma(z)\,\overline{g|_\sigma(z)}\, y^k\, d\mu(z)$$

$$= [\Gamma_0(q);\Gamma]^{-1} \int_{\mathcal{D}_{\Gamma'}} f(\sigma z)\overline{g(\sigma z)} (\mathrm{Im}\,\sigma z)^k \, d\mu(\sigma z).$$

$$= [\Gamma_0(q):\Gamma]^{-1} \int_{\sigma \mathcal{D}_{\Gamma'}} f(z)\overline{g(z)}\, y^k \, d\mu z.$$

Now $\sigma \mathcal{D}_{\Gamma'}$ is a fundamental domain for $\Gamma'' = \sigma\Gamma'\sigma^{-1}$

Choose Γ' small enough so $\Gamma'' \subseteq \Gamma_0(q)$.

Then $[\Gamma_0(q):\Gamma''] = [\Gamma_0(q) : \sigma\Gamma'\sigma^{-1}] = [\Gamma_0(q):\Gamma']$

So by Γ-invariance of the inner product, the proposition follows

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It follows that $\langle f|_\sigma, g\rangle = \langle f, g|_{\sigma^{-1}}\rangle$,

so $|_{\sigma^{-1}}$ is the adjoint of $|_\sigma$ with respect to the normalized $\langle,\rangle$.

Note $g|_{\sigma^{-1}} = g|_{\sigma'}$ where $\sigma\sigma' = \det\sigma$ (the adjugate matrix.)

Theorem If $(n,q)=1$, $f,g \in S_k(q,\chi)$ then

$$\langle T_n f, g\rangle = \chi(n)\langle f, T_n g\rangle$$

So the adjoint of T_n is $T_n^* = \bar{\chi}(n) T_n$ and T_n is normal.

Proof Sketch

$$\langle T_n f, g\rangle = n^{k/2-1} \sum_{\rho\in\Delta_n} \langle \bar{\chi}(\rho) f|_\rho, g\rangle$$

If $\rho = \begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$, $\rho' \in \begin{pmatrix} d & -b \\ 0 & a\end{pmatrix}$, so $\rho\rho' = n \cdot \mathrm{Id}$.

So $\langle f|_\rho, g\rangle = \langle f, g|_\rho\rangle$

So $\bar{\chi}(\rho) = \chi(a) = \chi(n)\bar{\chi}(d) = \chi(n)\chi(\rho')$

So $\langle \bar{\chi}(\rho) f|_\rho ; g\rangle = \chi(n) \langle f, \bar{\chi}(\rho'), g|_{\rho'}\rangle$

$\langle f, \bar{\chi}(\rho') g|_{\rho'}\rangle = \langle f, \bar{\chi}(\tau'\rho'\tau^{-1}) g|_{\tau'\rho'\tau^{-1}}\rangle \quad \forall \tau, \tau' \in \Gamma_0(q)$

<u>Exercise</u> Suppose $(a,d) \mid b$, then there exists $\tau, \tau' \in \Gamma_0(q)$ such that $\tau'\rho' = \tau\rho$.

Then $\langle \bar{\chi}(\rho) f|_\rho = g\rangle = \chi(n) \langle f, \bar{\chi}(\rho') g|_{\rho'}\rangle$

If n is square-free, we are done since $(a,d)=1$ automatically.

Summing over ρ, we prove the Theorem.

But $\mathbb{T}$ is commutative generated by T_p^χ, so we establish the result in general.

Let f be an eigenfunction of T_n.

$$T_n f = \lambda(n) f.$$

$$\langle T_n f, f\rangle = \lambda(n) \langle f, f\rangle = \chi(n) \langle f, \lambda(n) f\rangle$$

$$\Rightarrow \quad \lambda(n) = \chi(n)\bar{\lambda}(n) \text{ if } (n,q)=1.$$

<u>Theorem</u> There exists an orthonormal basis for $S_k(q,\chi)$ consisting of eigenfunctions of T_n for all $(n,q)=1$.

Newforms

Let $f \in S_k(q,\chi)$ be a Hecke eigenform.

That is, $T_n f = \lambda(n) f \qquad \forall (n,q)=1$

The condition $1=(n,q)$ causes many problems.

Suppose $f(z) = \sum_{m=1}^{\infty} a(m) e(mz)$

(since $a_n(m) = \sum_{d|(m,n)} \chi(d) d^{k-1} a(\frac{mn}{d^2})$)

we obtain: $\lambda(n)\, a(m) = \sum_{d|(m,n)} \chi(d) d^{k-1} a\left(\frac{mn}{d^2}\right)$

if $(n,q)=1$. If $m=1$ we have

$$a(n) = \lambda(n)\, a(1) \qquad \text{if } (n,q)=1.$$

However it is not the case that $a(1) \neq 0$, since we cannot control all of the $a(n)$.

There do exist f eigencusp form such that $a_f(1)=0$.

They come from "old forms", i.e. they come from χ mod q which are characters of some lower level q^*

Suppose χ is a Dirichlet character mod q, let q^* denote its conductor, i.e. $\chi(n)$ is also a dirichlet character mod q^* for $(n,q)=1$ and q^* is the minimal such q^*.

Let $q', d \in \mathbb{N}$ such that

$q^* | q'$, $q'd | q$ let χ' mod q' be χ restricted mod q'.

then

$$f(z) \in S_k(\Gamma_0(q'), \chi') \Rightarrow f(dz) \in S_k(\Gamma_0(q), \chi).$$

Indeed: $\begin{pmatrix} d & \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta d \\ \gamma/d & \delta \end{pmatrix}\begin{pmatrix} d & \\ & 1 \end{pmatrix}$

Then $f(dz) = \sum_{m \geq 1} a(m)\, e(mdz)$

$$= \sum_{m \equiv 0\, (d)} a(m/d)\, e(mz).$$

So $f(dz)$ has first coeff $= 0$.

Let $S_k^b(\Gamma_0(q), \chi)$ be the subspace of $S_k(\Gamma_0(q), \chi)$

Spanned by $f(dz)$, $d \mid q$,

$f \in S_k(\Gamma_0(q'), \chi')$ as above with $q' < q$
$q'd \mid q$.

Then let $S_k^\#(q, \chi)$ be the orthogonal complement.

So $S_k(q, \chi) = S_k^\#(q, \chi) \oplus S_k^b(q, \chi)$.

(orthogonal direct sum)

<u>Prop</u> $S_k^b, S_k^\#$ are stable under the Hecke operators with $(n, q) = 1$.

<u>Sketch</u> $T_n : S_k^b \to S_k^b$ can be checked directly on individual forms $f(dz)$. Orthogonality is preserved, since $T_n^* = \bar{\chi}(n) T_n$

So $T_n : S_k^\# \to S_k^\#$.

Thus each S_k^b, $S_k^\#$ has an orthonormal basis of eigenfcns of T_n, $(n,q)=1$.

The set $S_k^\#$ consists of newforms.

Having f be an eigenfcn of all T_n isn't quite adequate for studying eg. L-fcns.

~~If f is an eigenform of all (n,q)=1,~~

~~the~~ <u>PROP</u> Suppose f is a newform. Then

$$T_n f = \lambda(n) f \quad \text{for } \underline{\text{all}} \ n.$$

In particular, $a_f(1) \neq 0$ if f is a newform.
