Classical Modular Forms Lecture 17 30.11.2017

Let $Q \in \mathbb{Z}[x_1, \dots, x_n]$ be a quadratic form in r variables

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We are interested in studying

$$r(n,Q) = \#\{x \in \mathbb{Z}^r : n = Q(x)\} \text{ as } n \to \infty.$$

This uses crucially

$$\Theta(z,Q) = \sum_{m \in \mathbb{Z}^r} e(z\,Q(m)) = \sum_{n \geq 0} r(n,Q)\, e(nz)$$

We will show that $\Theta(z,Q)$ is a modular form for some congruence subgroup and nebentype char.

Let A be a real $r \times r$ matrix.

We say A is positive-definite if all its eigenvalues are positive.

Let $x \in \mathbb{R}^r$ a column vector and

$$x \cdot x = {}^{tr}x\, x = \sum_{j=1}^{r} x_j^2$$ be the standard inner product.

We have

$$A[x] = {}^{tr}x A x = \sum_{i,j} a_{ij} x_i x_j = \sum_i a_{ii} x_i^2 + 2\sum\sum_{i<j} a_{ij} x_i x_j.$$

is a quadratic form. In fact we have $Q(x) = \frac{1}{2}A[x]$ for some symmetric A.

We say $A[x]$ is even if all the coefficients are even.

If A is positive-definite, then so is its quadratic form.

We have: $\|x\|^2 \ll A[x] \ll \|x\|^2$ for all real x.

We have $\Theta(z, Q) = \sum_{m \in \mathbb{Z}^r} e\left(\frac{1}{2} A[m] z\right)$.

Clearly we have that $\Theta(z, Q)$ is holomorphic since the series converges absolutely and uniformly on compacta since $A[m] \gg |m|^2$.

We also will study

$$\Theta(z, x) = \sum_{m \in \mathbb{Z}^r} e\left(\frac{1}{2} A[m] z + B(x, m)\right)$$

for $x \in \mathbb{C}^r$, $B(x,y) = {}^{tr}x A y$ the bilinear form assoc. to A.

"Jacobi Θ-fcn"

Also, more important are the congruent Θ-fcns, $h \in \mathbb{Z}^r$.

$$\Theta(z; h) = \sum_{\substack{m \in \mathbb{Z}^r \\ m \equiv h\ (N)}} e\left(\frac{A[m]}{2N^2} z\right), \quad \text{note } \Theta(z;0) = \Theta(z).$$

Assume A has $\mathbb{Z}$-entries, and $N \in \mathbb{N}$ such that $A^* = NA^{-1}$ has $\mathbb{Z}$-entries. (clearly such N exists since $\det A = N$ suffices)

$\det A \cdot \det A^* = N^r$, so $N^r \equiv 0 \mod \det A$

i.e. every prime factor of $\det A$ divides N.

The automorphy of Θ, $\Theta(z,h)$ relies on the following ~~general~~ generalization of Poisson summation for Gaussians:

<u>Prop</u> Let A be a symmetric, pos-definite matrix. For $z \in \mathcal{H}$, and $x \in \mathbb{C}^r$ we have

$$\sum_m e\left(\frac{1}{2} A[m+x] z\right) = \frac{1}{(\det A)^{1/2}} \left(\frac{i}{z}\right)^{r/2} \sum_m e\left(\frac{-A^{-1}[m]}{2z} + {}^{tr}m x\right).$$

For example, if $x = 0$ then

$$\Theta(z,A) = (\det A)^{-1/2} \left(i/z\right)^{r/2} \Theta(-z^{-1}, A^{-1})$$

The proof is based on multivariable Poisson summation applied to a Gaussian:

$$\sum_{m \in \mathbb{Z}^r} f(m+x) = \sum_{m \in \mathbb{Z}^r} \hat{f}(m) e({}^{tr}m x).$$

We also need the following integral, which is a slight generalization of the familiar fact about Gaussians: $z \in \mathcal{H}$, $u \in \mathbb{R}$

$$\int_{\mathbb{R}} e\left(\tfrac{1}{2} y^2 z\right) e(-yu) dy = \left(\tfrac{i}{z}\right)^{1/2} e\left(\tfrac{-u^2}{2z}\right).$$

PROOF We apply Poisson Summation to

$$f(x) = e\left(\tfrac{1}{2} A[x] z\right)$$

To do this, we need to compute its Fourier transform.

So we diagonalize A:

Let U be an orthogonal matrix (${}^{tr}UU = Id$) so that

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} \text{ w/ } \lambda_1, \dots, \lambda_r \text{ are evals of } A.$$

Let $B = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_r} \end{pmatrix} U$, so that $A = {}^{tr}BB$.

To compute the Fourier transform, we make the change of vars $y = Bx$.

The goal is to compute

$$\hat{f}(v) = \int_{\mathbb{R}^r} e\left(\tfrac{1}{2}A[x]z\right) e(-{}^{tr}x v)\,dx.$$

We let $y = Bx$, $v = {}^{tr}Bu$, so $A[x] = {}^{tr}y\,y$

$$\downarrow$$

$${}^{tr}x v = {}^{tr}y u \qquad dx = \frac{dy}{\det B}.$$

Thus:
$$\hat{f}({}^{tr}Bu) = \frac{1}{\det B} \int_{\mathbb{R}^r} e\left(\tfrac{1}{2}{}^{tr}y y z\right) e(-{}^{tr}y u)\,dy$$

$$= \frac{1}{\det B} \prod_{j=1}^{r} \int_{\mathbb{R}} e\left(\tfrac{1}{2} y_j^2 z - y_j u_j\right) dy_j$$

$$= \frac{1}{\det B} \prod_{j=1}^{r} \left(\frac{i}{z}\right)^{1/2} e\left(-\frac{u_j^2}{2z}\right)$$

$$= \frac{1}{\det B} \left(\frac{i}{z}\right)^{r/2} e\left(-\frac{{}^{tr}u u}{2z}\right)$$

Going back to the original variables:

$${}^{tr}u u = {}^{tr}({}^{tr}B^{-1}v)\,{}^{tr}B^{-1} v = {}^{tr}v B^{-1}\,{}^{tr}B^{-1} v$$
$$= {}^{tr}v A^{-1} v = A^{-1}[v].$$

So $\hat{f}(v) = (\det(A))^{-1/2} \left(\frac{i}{z}\right)^{r/2} e\left(-\frac{A^{-1}[v]}{2z}\right)$

Hence
$$\sum_{m\in\mathbb{Z}^r} e\left(\tfrac{1}{2}A[m+x]z\right) = (\det A)^{-1/2} \left(\frac{i}{z}\right)^{r/2} \sum_{m\in\mathbb{Z}^r} e\left(\frac{-A^{-1}[m]}{2z} + {}^{tr}m x\right)$$

QED.

Note: If $\det A = 1$ then $\Theta(z, A^{-1}) = \Theta(z, A)$ since by changing $m \to Am$

~~tr...~~ ${}^{tr}m A^{-1} m \to {}^{tr}m\,{}^{tr}A A^{-1} A m = {}^{tr}m\,{}^{tr}A m$
$$= {}^{tr}m A m.$$

So $\Theta(-1/z, A) = (-iz)^{r/2}\,\Theta(z, A)$ in this special case.

Now we study the congruent Θ-fcn

$$\Theta(z;h) = \sum_{\substack{m \in \mathbb{Z}^r \\ m \equiv h\ (N)}} e\left(\frac{A[m]}{2N^2} z\right),$$ it depends only on h mod N.

In fact, we only need to study a subset of these. Let

$$\mathcal{H} = \{ h \ (\mathrm{mod}\, N) : Ah \equiv 0 \ (N) \}.$$

~~That~~ <u>Lemma</u> The set $\mathcal{H}$ is a finite abelian group with order
$$|\mathcal{H}| = \det A.$$

<u>PROOF</u> $A : \mathbb{R}^r \to \mathbb{R}^r$ transforms the unit box $[0,1]^r$ to a parallelotope P of volume $\det A$.

Since $A \in M_r(\mathbb{Z})$, the vertices of P have coordinates in $\mathbb{Z}^r$; so a standard combinatorial geometry fact says (exercise)

$$|P \cap \mathbb{Z}^r| = \mathrm{vol}\, P = \det A.$$

The matrix $N^{-1}A$ maps $[0,N]^r$ to P, and the integral points h in $[0,N]^r$ with $Ah \equiv 0\ (N)$ map to the integral vectors in P.

$\mathcal{H}$ is clearly a finite abelian group WRT addition. <u>QED</u>

We have $B(x,y) = {}^{tr}x A y$ is a non-degenerate bilinear form, and for any finite abelian group A, $A \cong \hat{A}$ (its dual).

For each $l \in \mathcal{H}$, $\psi(h,l) = e\left(\frac{B(h,l)}{N^2}\right)$ is a character of $\mathcal{H}$. There are $|\mathcal{H}|$ such characters, so all ~~the~~ characters are described in this way. (Note $Al \equiv 0(N)$, so if $N \mid h$, ${}^{tr}hAl$ is divisible by N^2)

We have the following orthogonality formula: 6/7

$$\sum_{h\in\mathcal{H}} \psi(h,l) = \begin{cases} |\mathcal{H}| = \det A & \text{if } l \equiv 0(N) \\ 0 & \text{else.} \end{cases}$$

<u>Prop</u> Let $h \in \mathcal{H}$. we have

$$\Theta(z+2,h) = e\left(\frac{A[h]}{N^2}\right)\Theta(z;h)$$

and if diag A is even, we have

$$\Theta(z+1,h) = e\left(\frac{A[h]}{2N^2}\right)\Theta(z,h).$$

<u>Proof</u> A is symmetric, so

$$A[x+y] = {}^{tr}x A(x+y) + {}^{tr}y A(x+y)$$
$$= {}^{tr}xAx + 2\,{}^{tr}xAy + {}^{tr}yAy$$

Let $m = h+nN$, we get

$$A[m] = A[h] + 2N\,{}^{tr}nAh + N^2A[n] \equiv A[h] \mod N^2$$

since $h \in \mathcal{H}$, so $Ah \equiv 0(N)$.

<u>If diag A is even</u>, then $\equiv A[h] \mod 2N$, since $A[n]$ even.

Now we establish the inversion formula for $\Theta(z;h)$.

<u>Prop</u> For any $h \in \mathcal{H}$, we have

$$\Theta(-1/z;h) = (\det A)^{-1/2}(-iz)^k \sum_{l\in\mathcal{H}} \psi(h,l)\,\Theta(z;l).$$

PROOF We apply today's first proposition with $x = hN^{-1}$

$$\text{LHS} = \sum_{m \in \mathbb{Z}^r} e\left(\tfrac{1}{2} A[m + hN^{-1}]z\right) = \Theta(z; h) = \sum_{\substack{m \\ m \equiv h (N)}} e\left(\frac{A[m]}{2N^2} z\right)$$

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$$\text{RHS} = \frac{1}{(\det A)^{1/2}} \left(\frac{i}{z}\right)^k \sum_{m \in \mathbb{Z}^r} e\left(-\frac{A^{-1}[m]}{2z} + {}^{tr}m h N^{-1}\right)$$

we change variables : $n = NA^{-1}m$

so ${}^{tr}m A^{-1} m = N^{-2}\, {}^{tr}n A n$.

$$= \frac{1}{(\det A)^{1/2}} \left(\frac{i}{z}\right)^k \sum_{\substack{n \in \mathbb{Z}^r \\ An \equiv 0 (N)}} e\left(\frac{-A[n]}{2N^2 z} + \frac{{}^{tr}n A h}{N^2}\right)$$

Change $z \longmapsto -1/z$

$$\Theta(-1/z, h) = (\det A)^{-1/2} (-iz) \sum_{An \equiv 0 (N)} e\left(\frac{A[n]}{2N^2} z\right) e\left(\frac{{}^{tr}n A h}{N^2}\right)$$

$$= (\det A)^{-1/2} (-iz) \sum_{l \in \mathcal{H}} \psi(h, l)\, \Theta(z, l)$$

QED