Classical Modular Forms Lecture 18 05.12.2017

Recall, A symmetric, positive-definite.

$\mathcal{H} = \{ h \in \mathbb{Z}^r \bmod N : Ah \equiv 0 (N) \}$ is a finite abelian group of cardinality $\det A$.

For each $\ell \in \mathcal{H}$, $\psi(h,\ell) = e\left(\frac{{}^{tr}h A \ell}{N^2} \right)$ is a character of $\mathcal{H}$
these are all the chars

We proved

$$\Theta\left(\frac{-1}{z} A, h\right) = (\det A)^{-1/2} (-iz)^{r/2} \sum_{\ell \in \mathcal{H}} \psi(h,\ell)\, \Theta(z, A, \ell)$$

This isn't enough to prove that Θ is modular.

Let $N \in \mathbb{N}$ be such that NA^{-1} has $\mathbb{Z}$-coefficients

Generalize to $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$ transformations

Assume $d \neq 0$ since otherwise $\tau = \pm \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and we are already done.

By swapping τ to $-\tau$ we can assume $d > 0$.

Suppose further that $b \equiv c \equiv 0 \ (2)$
OR $\operatorname{diag} A \equiv 0 (2)$.

Let's begin with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$

We have $\gamma z = \frac{b}{d} - \frac{1}{d(dz-c)}$, so

$$\Theta(\gamma z, A, h) = \sum_{m \equiv h (N)} e\left(\frac{A[m]}{2N^2} \left(\frac{b}{d} - \frac{1}{d(dz-c)} \right) \right) \text{ by definition}$$

Can simplify this

Suppose $m = h + n d N$ then

$$A[m] = A[h] + 2dN\, {}^{tr}n A h + d^2 N^2 A[n]$$

$\equiv A[h] \mod dN^2$

So under our additional assumptions, $e\left(\frac{bA[m]}{2dN^2}\right)$ only depends on $m \bmod dN$

So we split over residue classes:

$$\Theta(\gamma z, A, h) = \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{bA[g]}{2dN^2}\right) \underbrace{\sum_{m\equiv g(dN)} e\left(\frac{dA[m]}{2d^2N^2}\frac{-1}{dz-c}\right)}.$$

This is a Θ-fcn!

$$\Theta\left(\frac{-1}{dz-c}, dA, g\right) = \frac{1}{(\det dA)^{1/2}} (-i(dz-c))^{r/2} \sum_{l\in \mathcal{H}} \varphi(l,g)\Theta(dz-c, dA, l).$$

$$\underset{\text{expand}}{=} (\det dA)^{-1/2}(-i(dz-c))^{r/2} \sum_{\substack{l\in(\mathbb{Z}/dN)^r\\ dAl\equiv 0(Nd)}} e\left(\frac{{}^{tr}l A g}{(dN)^2}\right) \sum_{\substack{m\equiv l(dN)\\ m\in\mathbb{Z}^r}} e\left(\frac{dA[m]}{2d^2N^2}(dz-c)\right)$$

Collect all of the exponentials not depending on z above:

$$e\left(\frac{bA[g]}{2dN^2} + d\frac{{}^{tr}lAg}{(dN)^2} - \frac{cA[m]}{2dN^2}\right)$$

we have $m = l + dNn$, for some $n\in\mathbb{Z}^r$, so:

$$cA[m] = cA[l] + cdN\,{}^{tr}nAl + c(dN)^2A[n]$$
$$\equiv cA[l] \mod 2dN^2 \quad (\text{using the assumptions}).$$

So let

$$\varphi(h,l) = \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{bA[g] + 2\,{}^{tr}lAg - cA[l]}{2dN^2}\right)$$

This only depends on $h \bmod N$, but I claim that it also only depends on $l \bmod N$ (a priori: $l \bmod dN$)

We have $\Theta(\gamma_z, Ah) = (\det dA)^{-1/2}(i(c-dz))^{r/2} \sum_{\substack{l \bmod dN \\ Al \equiv 0 (N)}} \varphi(h,l) \sum_{\substack{m \equiv l (dN) \\ m \in \mathbb{Z}^r}} e\left(\frac{A[m]}{2N^2} z\right).$

We can further simplify $\varphi(h,l)$ and prove the claim.

Change variable $g \to g + cl$, so g runs mod dN, st $g \equiv h - cl (N)$

Numerator of $\varphi(h,l)$:

$$\begin{aligned} &bA[g] + 2bc\,{}^{tr}lAg + bc^2A[l] \\ &\quad + 2\,{}^{tr}lAg + 2cA[l] \\ &\qquad - cA[l] \end{aligned} = \begin{aligned} &bA[g] + (2(ad-bc)+2bc)\,{}^{tr}lAg \\ &\quad + c(bc+1)A[l] \end{aligned}$$

$$= bA[g] + 2ad\,{}^{tr}lAg + acdA[l]$$

So $g = h - cl\ (N)$ thus

$$2ad\,{}^{tr}lAg = 2ad\,{}^{tr}lAh - 2acdA[l] \bmod N^2$$

Thus: $\varphi(h,l) = e\left(\frac{2a\,{}^{tr}lAh - acA[l]}{2N^2}\right)\varphi(h-cl,0)$

$$= e\left(-\frac{ac\,A[l]}{2N^2}\right)\psi(ah,l)\,\varphi(h-cl,0),$$

so it only depends on l mod N.

Thus: $\Theta(\gamma_z, A, h) = \left(\frac{i(c-dz)}{d}\right)^{r/2}(\det A)^{-1/2}\sum_{l \in \mathcal{H}} \varphi(h,l)\,\Theta(z, A, l)$

Change z to $\frac{-1}{z}$, so $\gamma(-\frac{1}{z}) = \tau(z)$.

$$\Theta(\tau z, A, h) = \left(\frac{i(c+d/z)}{d}\right)^{r/2}(\det A)^{-1/2}\sum_{l \in \mathcal{H}} \varphi(h,l)\,\Theta(-\tfrac{1}{z}, A, l).$$

We apply the Prop again! Note $\left(\frac{i}{z}(cz+d)\right)^{r/2}\left(\frac{z}{i}\right)^{r/2} = (cz+d)^{r/2}$

Thus:

$$\boxed{\Theta(\tau z, A, h) = \left(\frac{cz+d}{d}\right)^{r/2} \frac{1}{\det A} \sum_{\ell \in \mathcal{H}} \Phi(h,\ell)\, \theta(z,A,\ell) \quad \text{where } \Phi(h,\ell) = \sum_{h' \in \mathcal{H}} \varphi(h,h')\psi(h',\ell)}$$

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Note: This already implies the growth condition:

Let $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$, $\sigma_{\mathfrak{a}} \in SL_2\mathbb{Z}$

$$\Theta|_{\sigma_{\mathfrak{a}}}(z,A,h) = j_{1/2}(z,\gamma)^r\, \Theta(\sigma_{\mathfrak{a}} z, A, h)$$

Fourier expansion at $\mathfrak{a}$ of Θ is the Fourier expansion of $\Theta|_{\sigma_{\mathfrak{a}}}$ at ∞: So the above boxed formula, and fact that $\theta(z,A,l)$ has no negative Fourier coefficients $\Rightarrow$ growth cond.

(See lecture 7) There are only finitely many cusps.

We can be more explicit. Let us make one of two assumptions:

(1) $c \equiv 0\ (2N)$

(2) $c \equiv 0\ (N)$ and $\operatorname{diag} NA^{-1} \equiv 0\ (2)$.

Note $\operatorname{diag} NA^{-1} \equiv 0(2)$ and $Al \equiv 0\ (N) \Rightarrow A[l] \equiv 0\ (2N)$.

Why? let $m = N^{-1}Al \in \mathbb{Z}^r$, then

$$N^{-1}A[l] = {}^{tr}l\,m = N\, {}^{tr}m A^{-1} m \equiv 0\ (2)$$

Thus if one of the above two conditions holds, we have

$$\varphi(h,l) = \psi(ah,l)\,\varphi(h,0)$$

Thus $\Phi(h,l) = \varphi(h,0) \sum_{h' \in \mathcal{H}} \psi(ah,h')\psi(h',l)$

$$= \varphi(h,0)\begin{cases} \det A & \text{if } \ell \equiv -ah\ (N) \\ 0 & \text{else.} \end{cases}$$

So the ~~above~~ sum over ℓ in the above boxed eqn has only one term! (since $\Theta(z,h,A) = \Theta(z,A,-h)$)

$$\Theta(\tau z; A, h) = \varphi(h,0)\, d^{-r/2}(cz+d)^{r/2}\,\Theta(z, A, ah).$$

Note: $\displaystyle \varphi(h,0) = \sum_{\substack{g\,(dN) \\ g \equiv h\,(N)}} e\left(\frac{bA[g]}{2dN^2}\right)$ is a Gauss sum (see Lecture 2, exos 1.12, 1.15)

Lemma $\displaystyle \varphi(h,0) = e\left(\frac{abA[h]}{2N^2}\right)\left(\frac{\det A}{d}\right)\left(\varepsilon_d^{-1}\left(\frac{2c}{d}\right)\sqrt{a}\right)^r$

Where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol (Recall from lecture 2)

$$\varepsilon_d = \begin{cases} 1 & d \equiv 1\ (4) \\ i & d \equiv 3\ (4) \end{cases} \qquad \text{Recall } j_{1/2}(z,\gamma) = \varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}$$

PROOF (Omitted/Exercise).

Except for this omitted proof, we have a complete proof of:

PROPOSITION Let $\tau = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2\mathbb{Z}$ w/ $d \equiv 1\ (2)$

Suppose that one of the following holds:

1. $c \equiv 0\ (2N)$ and $b \equiv 0\ (2)$
2. $c \equiv 0\ (2N)$ and $\operatorname{diag} A \equiv 0\ (2)$
3. $c \equiv 0\ (2N)$, $\operatorname{diag} A \equiv \operatorname{diag} NA^{-1} \equiv 0\ (2)$.

Then for any $h \in \mathcal{H}$

$$\Theta(\tau z, A, h) = e\left(\frac{ab\,A[h]}{2N^2}\right)\left(\frac{\det A}{d}\right)\left(\varepsilon_d^{-1}\left(\frac{2c}{d}\right)(cz+d)^{1/2}\right)^r \Theta(z, A, ah)$$

<u>COR 1</u> Let $h \in \mathcal{H}$, $\Theta(z,h)$ is a modular form for $\Gamma(4N)$ of weight $r/2$, character $\left(\frac{2}{d}\right)^r$.

<u>Indeed</u> $d \equiv 1 (4N)$ and $N \mid \det A \det A^*$

So $\left(\frac{\det A}{d}\right) = 1$. $\quad A^* = NA^{-1}$

Note $\Theta(z,A) = \Theta(z,A;0)$.

Let diag A be even, then

$$\Theta(z,A) \in M_{r/2}\left(\Gamma_0(2N), \left(\frac{\det A}{d}\right)\left(\frac{2}{d}\right)^r\right).$$

Suppose r is <u>even</u> then $\varepsilon_d^2 = \left(\frac{-1}{d}\right)$

let $D = (-1)^{r/2}\det A$.

<u>COR</u> Let diag $A \equiv$ diag $NA^{-1} \equiv 0(2)$.

Then $\Theta(z,A) \in M_{r/2}\left(\Gamma_0(N), \left(\frac{D}{d}\right)\right)$.

<u>PROOF</u> We use the 3rd option but still need to treat d even separately.

If $2 \mid d$ then $2 \nmid c$ and we study $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d\end{pmatrix}$

Example. $N=1$ then recall $\Theta\left(\frac{-1}{z}, A\right) = (-iz)^{r/2}\,\Theta(z,A)$

So since $(-i)^{r/2} = 1$, we must have $r \equiv 0\ (8)$.

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<u>Minkowski</u>:

$$A = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 4 & 3 & & & & \\ & & 3 & 4 & 5 & & & \\ & & & 5 & 20 & 3 & & \\ & & & & 3 & 12 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}$$

Then $\Theta(z,A) \in M_4(SL_2\mathbb{Z})$

So $\Theta(z,A) = E_4(z)$

so $r(n, \tfrac{1}{2}A[x]) = 240\,\sigma_3(n)$.